18th Benelux Mathematical Olympiad

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Problems and solutions

Problem Selection Committee
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Problem 1

Let n be a positive integer and let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers such that

$$a_k(b_1 + \cdots + b_k) \leqslant k \qquad \text{and} \qquad b_k(a_1 + \cdots + a_k) \leqslant k$$

for $k = 1, 2, \ldots, n$. Prove that $(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) \leqslant n^2$.

Solution 1.
Define $A_k = a_1 + a_2 + \ldots + a_k$ and $B_k = b_1 + b_2 + \ldots + b_k$. Then we are given that $a_k B_k \leqslant k$ and $b_k A_k \leqslant k$ for all k, and we are asked to show that $A_n B_n \leqslant n^2$. We will prove this statement by induction on n. The base case $n = 1$ follows immediately from the given condition for $k = 1$. Now we assume the statement for some $k = \ell \geqslant 1$, meaning that $A_\ell B_\ell \leqslant \ell^2$. If $a_{\ell+1} b_{\ell+1} \geqslant 1$, then using the given inequalities for $k = \ell + 1$ yields

$$(\ell + 1)^2 \geqslant a_{\ell+1} B_{\ell+1} \cdot b_{\ell+1} A_{\ell+1} = a_{\ell+1} b_{\ell+1} \cdot A_{\ell+1} B_{\ell+1} \geqslant A_{\ell+1} B_{\ell+1},$$

thereby immediately completing the inductive step. If, on the other hand, $a_{\ell+1} b_{\ell+1} \leqslant 1$, then we derive using the inductive hypothesis that

$$a_{\ell+1} B_\ell \cdot b_{\ell+1} A_\ell = a_{\ell+1} b_{\ell+1} \cdot A_\ell B_\ell \leqslant 1 \cdot \ell^2 = \ell^2,$$

which implies that $a_{\ell+1} B_\ell \leqslant \ell$ or $b_{\ell+1} A_\ell \leqslant \ell$. Without loss of generality, we assume that $b_{\ell+1} A_\ell \leqslant \ell$. From the given condition for $k = \ell + 1$ we also know that $a_{\ell+1} B_{\ell+1} \leqslant \ell + 1$. Combining the estimates $A_\ell b_{\ell+1} \leqslant \ell$ and $a_{\ell+1} B_{\ell+1} \leqslant \ell + 1$ with the inductive hypothesis $A_\ell B_\ell \leqslant \ell^2$, we now find

$$A_{\ell+1} B_{\ell+1} = A_\ell B_\ell + A_\ell b_{\ell+1} + a_{\ell+1} B_{\ell+1} \leqslant \ell^2 + \ell + (\ell + 1) \leqslant (\ell + 1)^2,$$

thereby completing the induction. □

Solution 2.
We also do induction using the same notation as solution 1, the induction basis again being trivial. By the induction hypothesis we can assume that $A_\ell B_\ell \leqslant l^2$. Suppose first that $(\ell + 1) a_{\ell+1} \geqslant A_{\ell+1}$ (meaning $a_{\ell+1}$ is at least the average of all previous terms). The given condition for $k = \ell + 1$ gives $\ell + 1 \geqslant a_{\ell+1} B_{\ell+1}$, so

$$(\ell + 1)^2 \geqslant (\ell + 1) a_{\ell+1} B_{\ell+1} \geqslant A_{\ell+1} B_{\ell+1}.$$

Analogously we are done when $(\ell+1) b_{\ell+1} \geqslant B_{\ell+1}$, so we can assume $(\ell+1) a_{\ell+1} \leqslant A_{\ell+1}$ and $(\ell+1) b_{\ell+1} \leqslant B_{\ell+1}$. From this we get $\ell a_{\ell+1} \leqslant A_{l+1} - a_{\ell+1} = A_\ell$ and analogously $\ell b_{\ell+1} \leqslant B_\ell$. Putting these inequalities and the induction hypothesis together gives

$$A_{\ell+1} B_{\ell+1} = (A_\ell + a_{\ell+1})(B_\ell + b_{\ell+1}) \leqslant (A_\ell + \frac{A_\ell}{\ell})(B_\ell + \frac{B_\ell}{\ell}) = A_\ell B_\ell (1 + \frac{1}{\ell})^2 \leqslant \ell^2 (1 + \frac{1}{\ell})^2 = (\ell+1)^2,$$

completing the induction. $\square$

Solution 3.
Let $A_k = a_1 + ... + a_k$ and $B_k = b_1 + ... + b_k$, with $A_0 = B_0 = 0$. Note that $A_k, B_k > 0$ for all $k > 0$. The given inequalities rewrite to

$$(A_k - A_{k-1})B_k \leq k \text{ and } (B_k - B_{k-1})A_k \leq k.$$

We have to prove that $A_kB_k \leq k^2$ for all $k > 0$. For $k = 1$ we then get $A_1B_1 \leq 1$. We proceed by induction on k. Assume, therefore, that $k > 1$ and that we know $A_{k-1}B_{k-1} \leq (k-1)^2$. We can then rewrite the above inequalities as

$$A_k - \frac{k}{B_k} \leq A_{k-1} \text{ and } B_k - \frac{k}{A_k} \leq B_{k-1}.$$

Now if $A_kB_k \leq k$, we are done; hence we assume $A_kB_k > k$. In that case, the above inequalities imply that

$$\left(A_k - \frac{k}{B_k}\right)\left(B_k - \frac{k}{A_k}\right) \leq A_{k-1}B_{k-1} \leq (k-1)^2.$$

This then implies that

$$A_kB_k + \frac{k^2}{A_kB_k} \leq k^2 + 1,$$

which rewrites to

$$0 \leq \frac{1}{A_kB_k}(A_kB_k - k^2)(1 - A_kB_k),$$

implying that $1 \leq A_kB_k \leq k^2$, as desired. $\square$

Problem 2

Let a_0 and N be positive integers. Ben and Lux take turns playing the following game: At the start of the game, the number N is written on a board. At move $i \geqslant 1$ of the game, the player whose turn it is chooses a positive integer $a_i \leqslant a_{i-1}$, and replaces the number n on the board at that point with $n - a_i$. The first player to write down a number $n \leqslant 0$ loses the game. Ben makes the first move. For each $N \geqslant 2$, determine the smallest possible value of a_0 for which Ben can win the game irrespectively of how Lux plays.

Solution 1.
The smallest possible a_0 for which Ben can win the game is the largest power of 2 that divides $N - 1$. We prove the following more general claim:

Claim. Lux has a winning strategy for the game (N, a_0) if and only if $N - 1$ is divisible by 2^k, where k is such that $2^k \leqslant 2a_0 < 2^{k+1}$.

We first prove that if $N \equiv 1 \mod 2^k$ (where k is such that $2^k \leqslant 2a_0 \leqslant 2^{k+1}$), then Lux has a winning strategy. We do so by induction on a_0. For $a_0 = 1$ the players take turns subtracting 1, so indeed Lux wins if $N \equiv 1 \mod 2$. Now consider $a_0 = m > 1$ and suppose that Lux indeed has a winning strategy for $N \equiv 1 \mod 2^k$ (where k is such that $2^k \leqslant 2a_0 \leqslant 2^{k+1}$) for all $a_0 < m$. Let k be such that $2^k \leqslant 2m < 2^{k+1}$. By induction hypothesis we already know that Lux has a winning strategy for the game $(N, m - 1)$. If Ben chooses a number $a < m$ in their first move, then the game follows the rules of the game $(N, m - 1)$, and so we know that Lux has a winning strategy. Therefore, we only have to consider the case where Ben starts by choosing the number $a = m$. In this case, Lux subtracts $2^k - m \leqslant m$ on their first move, and the players end up playing the game $(N - 2^k, 2^k - m)$ after the first two moves. If $2^k - m < m$, then Lux can win this game thanks to the induction hypothesis. If $2^k - m = m$, then we can continue applying the same logic until we end up in the game $(1, m)$ (after all, $N \equiv 1 \mod 2^k$), which is also won by Lux. So Lux indeed wins for all the games (N, t) where $N \equiv 1 \mod 2^k$ (where k is such that $2^k \leqslant 2a_0 \leqslant 2^{k+1}$).

We now show that Ben wins the game for all other N. We again use induction on a_0, noting that Ben indeed wins for even N when $a_0 = 1$. Now suppose that $a_0 = m > 1$ and we have shown that Ben indeed has a winning strategy for $N \not\equiv 1 \mod 2^k$ (where k is such that $2^k \leqslant 2a_0 \leqslant 2^{k+1}$) for all $a_0 < m$. Let k be such that $2^k \leqslant 2m < 2^{k+1}$, we show that Ben has a winning strategy when $N \not\equiv 1 \mod 2^k$. If $2^{k-1} < m$ or $N \not\equiv 1 \mod 2^{k-1}$, then we know by the induction hypothesis that Ben has a winning strategy for the game $(N, m - 1)$. This strategy also allows Ben to win the game (N, m). So the only situation we have to consider is the case where $m = 2^{k-1}$ and $N \equiv 1 \mod 2^{k-1}$. In this case Ben chooses in their first move the number $a = 2^{k-1}$. Then after the first move, the game turns into the game $(N - 2^{k-1}, 2^{k-1})$, where $N - 2^{k-1} \equiv 1 \mod 2^k$ (with the players' roles reversed). We already established above that the second player has a winning strategy for this game. So this game can also be won by Ben, and the proof is complete. □

Solution 2.
Let $N' = N - 1$ and write N' together with all a_i in binary. We claim that all losing positions are those where N' ends in at least as many zero's as a_{i-1} has digits. If this is the case let's call it an L-state and if not call it a W-state. We will show that from a W-state it is possible to go to an L-state and from an L-state any move will land you in a W-state. Together with the fact that for $N' = 0$ all positions are losing and indeed also L-states, this proves our claim (as the game is clearly finite).

Suppose we are in a W-state, so N' ends in fewer zero's then a_{i-1} has digits. Take a_i to be the power of two that ends in as many zero's as N'. This is allowed as N' ends in fewer zero's then a_{i-1} has digits, so $a_i \leqslant a_{i-1}$. Now we will show that $(N' - a_i, a_i)$ is an L-state. We see that $N' - a_i$ ends in at least one more zero then N', while a_i has exactly one more digit then the amount of zero's that N' ends in. So indeed $N' - a_i$ ends in at least as many zero's as a_i has digits and we end op in an L-state.

Now suppose we are in an L-state, so N' ends in at least as many zero's as a_{i-1} has digits. We need to show that for any $a \leqslant a_{i-1}$ we have that $(N' - a, a)$ is a W-state. Suppose by contradiction there is an a_i such that $N' - a_i$ ends in as least as many zero's as a_i has digits. Then the last digits of $(N' - a_i) + a_i = N'$ should be a copy of a_i, but at the same time N' should end in as least as many zero's as a_i has digits (as $a_i \leqslant a_{i-1}$), a contradiction.

Now with our claim we see indeed that the smallest possible a_0 for which Ben can win the game is the largest power of 2 that divides $N - 1$.

□

Problem 3

Let $ABCD$ be a parallelogram such that $|AC| = |BC|$, and let A' be the reflection of A in B. Lines BD and $A'D$ meet the circumcircle of triangle ACD again at E and F, respectively. Lines AE and $A'C$ intersect at G. Show that lines AC, EF, BG are concurrent or pairwise parallel.

Solution 1.
First of all note that $|A'B| = |AB| = |CD|$ and $A'B \parallel CD$, so $A'BDC$ is a parallelogram. From $\angle BAC = \angle ACD = \angle CAD$ it follows that BA is tangent to (ACD). Then by

$$\angle BCG = \angle BCA' = \angle DBC = \angle ADB = \angle ADE = \angle BAE = \angle BAG$$

it follows that $ABGC$ is cyclic. Now let P be the projection of D onto AB. Then $|AB| = |A'B| = 2|AP|$. We can use Pythagoras to get that

$$|A'D|^2 - |BD|^2 = |A'P|^2 + |DP|^2 - |BP|^2 - |DP|^2 = 25|AP|^2 - 9|AP|^2 = 16|AP|^2 = |A'A|^2.$$

Which can be rewritten to

$$|BD|^2 = |A'D|^2 - |A'A|^2 = |A'D|^2 - |A'D||A'F| = |A'D|(|A'D| - |A'F|) = |A'D||DF|.$$

By the power of a point theorem this implies DB is tangent to $(A'BF)$. It follows that

$$\angle DFB = 180° - \angle A'FB = \angle DBA' = 180° - \angle ABD = 180° - \angle EDC = \angle EFC,$$

implying $\angle CFD = \angle CFE - \angle EFD = \angle BFD - \angle EFD = \angle BFE$. Using this we get

$$\angle BGE = \angle BGA = \angle BCA = \angle CAD = \angle CFD = \angle BFE,$$

which implies that $BGFE$ is cyclic. Now we know that $BGFE$, $AEFC$ and $ABGC$ are all cyclic so by the radical axis theorem AC, EF, BG either concur or are pairwise parallel and we are done. □

Solution 2.
We also start by noticing that $A'BDC$ is a parallelogram and prove that $ABGC$ is cyclic:

$$\angle AGC = 180° - \angle ACG - \angle GAC = 180° - \angle ACB - \angle BCA' - \angle EAC =$$

$$180° - \angle DAC - \angle ADB - \angle BDC = \angle DCA = \angle CBA.$$

Now let N and M be the midpoints of AC and BC respectively (which are also the midpoints of BD and $A'D$ respectively). We have that

$$\angle AFM = \angle AFD = \angle ACD = \angle ABC = \angle ABM,$$

so $ABFM$ is cyclic. By symmetry we know that $ABNM$ is cyclic, so also $ABFMN$ is cyclic. First notice that

$$\angle CAM = \angle NAM = \angle NBM = \angle DBC = \angle ADB = \angle BCA' = \angle BCG = \angle BAG = \angle BAE.$$

Now we see that

$$\angle EFB = \angle DFB - \angle DFE = (180^\circ - \angle MAB) - (180^\circ - \angle DAE) = \angle DAE - \angle MAB =$$

$$\angle DAM - \angle EAB = \angle DAC + \angle CAM - \angle EAB = \angle DAC = \angle ACB = \angle AGB,$$

so we find that $BGFE$ is cyclic. Again we are done by the radical axis theorem. □

Solution 3.
Let N be as in the previous solution, and notice that NE is a midline of $\triangle AGC$ as N is the middle of AC and NE is parallel to CG (in particular, E is the middle of AG). We start again by proving that $ABGC$ is cyclic:

$$\angle AGC = \angle AEN = \angle AED = \angle ACD = \angle CAB = \angle ABC.$$

Now we see that

$$\angle FAE = \angle FDE = \angle A'DB = \angle DA'C = \angle FA'G,$$

so $AA'GF$ is cyclic. We also see that

$$\angle NAE = \angle CAE = \angle CDE = \angle CDB = \angle BA'C = \angle AA'G = \angle ABE,$$

so NA is tangent to both (ABE) and $(AA'GF)$. Just like in the first solution we know that AA' is tangent to $(AEFCD)$, which we can use to get

$$\angle NCF = \angle ACF = \angle ADF = \angle FAA' = 180^\circ - \angle FGA' = \angle FGC,$$

so NC is tangent to (CGF). Now as $|AN| = |CN|$ we see that the power of N to $(AA'GF)$ and (CGF) is the same, so N lies on the radical axis of those circles. So N, F and G are collinear. This means that $NE \cdot NB = AN^2 = CN^2 = NF \cdot NG$, so we get that $BGFE$ is cyclic. Again we are done by the radical axis theorem. □

Solution 4.
We provide another proof for the fact that $BGFE$ is cyclic assuming that $ABGC$ and $AA'GF$ are cyclic. We have that

$$\angle CEF = \angle CDF = \angle CDA' = \angle DA'A = \angle FA'A = \angle FGA = \angle FGE,$$

so CE is tangent to (FEG). Notice that $\angle GEB = \angle AED = \angle ACD = \angle CAB$ and $\angle EGB = \angle AGB = \angle ACB$, so $\triangle EGB \sim \triangle ACB$. From this we derive that

$$\angle CEG = 180^\circ - \angle AEC = \angle ADC = \angle ABC = \angle EBG,$$

so CE is also tangent to (EGB). It follows that $BGFE$ is cyclic. □

Solution 5.
We provide yet another proof for the fact that $BGFE$ is cyclic assuming that $ABGC$ and $AA'GF$ are cyclic. Firstly, as $ABGC$ is cyclic we know that there is a reflexional homothety around A' taking $\triangle A'GB$ to $\triangle A'AC$. As $\angle A'BD = \angle A'CD$ the line BE will go to the line CD and as $\angle EGB = \angle ACB = \angle DAC$ the line GE will go to AD. Thus there intersections, E resp. D, will get mapped to each other and $\triangle BGE \sim \triangle CAD$. Furthermore, we know that $\angle FGE = \angle FA'A = \angle AA'D$ (as $AA'GF$ is cyclic) and $\angle FEG = 180^\circ - \angle AEF = \angle ADF$, so $\triangle ADA' \sim \triangle FEG$. Putting this together we get

$$\angle EFG + \angle GBE = \angle A'AD + \angle ACD = \angle BAD + \angle ADC = 180^\circ,$$

so $BGFE$ is cyclic. □

Solution 6.
We provide another proof of the problem assuming that $ABGC$ and $AA'GF$ are cyclic. Let S be the intersection of DA and BG and let T be the other intersection point of BC with $(AEFCD)$. Then we have that

$$\angle SGA' = \angle BGA' = 180^\circ - \angle CGB = \angle CAB = \angle CBA = \angle BAS = \angle SAA',$$

so S lies on $(AA'GF)$. Furthermore

$$\angle DFS = 180^\circ - \angle FDS - \angle ASF = 180^\circ - \angle A'DA - \angle AA'F = 180^\circ - \angle A'DA - \angle AA'D =$$

$$\angle DAA' = \angle DAB = \angle BCD = \angle TCD = \angle TFD,$$

so S, T and F are collinear. Now by Pascal on $ADEFTC$ we get that S, B and the intersection of AC with EF are collinear, so we are done as S lies on BG. □

Solution 7.
We provide another proof of the fact that $BGFE$ is cyclic assuming that $ABGC$ is cyclic. Let U be the second intersection of $A'C$ with (ACD). Then $DEUC$ is an isosceles trapezoid

(as $DE||UC$), so $|EU| = |DC| = |AB| = |A'B|$. That means that $A'BEU$ is an isosceles trapezoid as well, so its cyclic. Also, as AB is tangent to (ACD) we know that $BD \cdot BE = BA^2 = A'B^2$, so BA' is tangent to $(A'DE)$. Then we see that $\angle BUE = \angle BA'E = \angle A'DE = \angle FDE = \angle FUE$, so B, F and U are collinear. Now we calculate

$$\angle BFE = 180° - \angle EFU = \angle UCE = \angle CUD = \angle CAD = \angle ACB = \angle AGB = \angle EGB,$$

so $BGFE$ is cyclic.

□

Problem 4

Find all pairs (a, b) of positive integers for which there exists a positive integer c such that $a^2 + b^2 + c^2$ divides $(a + b + c)^2$.

Solution.
The answer is all pairs (a, b) such that ab is a square.

We first notice that

$$a^2 + b^2 + c^2 < (a + b + c)^2 \leqslant 3(a^2 + b^2 + c^2),$$

where the second inequality follows from $(a - b)^2 + (b - c)^2 + (c - a)^2 \geqslant 0$. This tells us that $a^2 + b^2 + c^2 \mid (a + b + c)^2$ can only happen if $(a + b + c)^2 = 2(a^2 + b^2 + c^2)$ or $(a+b+c)^2 = 3(a^2+b^2+c^2)$. If $(a+b+c)^2 = 3(a^2+b^2+c^2)$, then we must have equality in $(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$, which only occurs if $a = b = c$. This shows that all (a, b) with $a = b$ are solutions, and those are indeed of the described form. Any other solutions must correspond to $(a+b+c)^2 = 2(a^2+b^2+c^2)$, or, equivalently, $a^2+b^2+c^2 = 2ab+2bc+2ca$. This equation rewrites to $(c - a - b)^2 = 4ab$. This shows that (a, b) can only be a solution if $4ab$, and therefore ab, is a square. On the other hand, all such (a, b) are indeed solutions, because if $ab = x^2$, then $c = a + b + 2x$ works. □

Remark 1. Instead of rewriting the equation $(a+b+c)^2 = 2(a^2+b^2+c^2)$ to $(c-a-b)^2 = 4ab$, one could also look at the discriminant of $(a + b + c)^2 = 2(a^2 + b^2 + c^2)$ as a function of c. This gives that $16ab$ has to be a square, so ab has to be as well.

Remark 2. The number of solutions c is always equal to 2. If $a \neq b$, then the solutions are given by $c = a + b \pm 2\sqrt{ab}$ (note that $a + b - 2\sqrt{ab}$ is positive by AM-GM), both of which give a solution with $(a + b + c)^2 = 2(a^2 + b^2 + c^2)$. If $a = b$, then there is only one c such that $(a+b+c)^2 = 2(a^2+b^2+c^2)$ (the other solution is 0), but there is the additional solution $c = a = b$ with $(a + b + c)^2 = 3(a^2 + b^2 + c^2)$.

Remark 3. All primitive solutions are given by $(1, 1, 1)$ and $(n^2, m^2, (n+m)^2)$ for $n, m \in \mathbb{Z}$ with n and m coprime (so a general solution will be a scalar multiple of one of these).