16th Benelux Mathematical Olympiad

Valkenswaard, 26th – 28th April 2024

Problems and Solutions

Problem Selection Committee

Ward van der Schoot (Chair)

Tijs Buggenhout, Justin Vast (Belgium)

Kevin van Dijk, Wietze Koops (the Netherlands)

Bernard Felten, Pierre Haas (Luxembourg)

Problem 1

(a) Let $a_0, a_1, \ldots, a_{2024}$ be real numbers such that $|a_{i+1} - a_i| \leqslant 1$ for $i = 0, 1, \ldots, 2023$. Find the minimum possible value of

$$a_0a_1 + a_1a_2 + \cdots + a_{2023}a_{2024}.$$

(b) Does there exist a real number C such that

$$a_0a_1 - a_1a_2 + a_2a_3 - a_3a_4 + \cdots + a_{2022}a_{2023} - a_{2023}a_{2024} \geqslant C$$

for all real numbers $a_0, a_1, \ldots, a_{2024}$ such that $|a_{i+1} - a_i| \leqslant 1$ for $i = 0, 1, \ldots, 2023$?

Solution 1

(a) The minimum value is -506. Note that from $|a_i - a_{i-1}| \leq 1$ it follows that

$$a_ia_{i-1} = \frac{(a_i + a_{i-1})^2 - (a_i - a_{i-1})^2}{4} \geq -\frac{(a_i - a_{i-1})^2}{4} \geq -\frac{1}{4}.$$

Adding this for $i = 1, 2, \ldots, 2024$, we obtain that

$$a_0a_1 + a_1a_2 + a_2a_3 + \ldots + a_{2023}a_{2024} \geq 2024 \cdot -\frac{1}{4} = -506.$$

We now show that this value can be attained. Indeed, for the sequence $(a_0, a_1, \ldots, a_{2024}) = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ with alternating $\frac{1}{2}$'s and $-\frac{1}{2}$'s, each term a_ia_{i-1} is equal to $-\frac{1}{4}$, leading to $a_0a_1 + a_1a_2 + a_2a_3 + \ldots + a_{2023}a_{2024} = 2024 \cdot -\frac{1}{4} = -506$. □

(b) No, such a C does not exist. We argue by contradiction. Suppose C has this property, and consider the sequence defined by $a_0 = C$ and $a_i = C - 1$ for $i = 1, 2, \ldots, 2024$ satisfies the condition in the problem. For this sequence, we have $a_ia_{i+1} - a_{i+1}a_{i+2} = 0$ for $i = 2, 4, \ldots, 2022$, so the sum

$$a_0a_1 - a_1a_2 + a_2a_3 - a_3a_4 + a_4a_5 - a_5a_6 + \ldots + a_{2022}a_{2023} - a_{2023}a_{2024}$$

is equal to

$$a_0a_1 - a_1a_2 = C(C-1) - (C-1)^2 = C - 1 < C,$$

contradiction. □

Solution 2

We give an alternative construction for part (b). We choose a real constant N, from which we define $a_i = N + i$ for each $i = 0, 1, \ldots, 2024$, which clearly satisfies the requirement $|a_i - a_{i-1}| \leq 1$ for each $i = 0, 1, \ldots, 1011$. Then, it can be seen that for each $i = 0, 1, \ldots, 1011$ that

$$a_{2i}a_{2i+1} - a_{2i+1}a_{2i+2} = a_{2i+1}(a_{2i} - a_{2i+2}) = -2(N + 2i + 1) \leqslant -2N.$$

From this, it can be concluded that

$$a_0a_1 - a_1a_2 + a_2a_3 - a_3a_4 + a_4a_5 - a_5a_6 + \ldots + a_{2022}a_{2023} - a_{2023}a_{2024} \leqslant 1012 \cdot -2N = -2024N.$$

As N is a constant which can be arbitrarily chosen, there is no constant C which lower bounds the given expression. □

Problem 2

Let n be a positive integer. In a coordinate grid, a *path* from $(0,0)$ to $(2n,2n)$ consists of $4n$ consecutive unit steps $(1,0)$ or $(0,1)$. Prove that the number of paths that divide the square with vertices $(0,0)$, $(2n,0)$, $(2n,2n)$, $(0,2n)$ into two regions with even areas is

$$\frac{\binom{4n}{2n}+\binom{2n}{n}}{2}.$$

Solution 1

Let X denote the set of paths for which A and B have even area and let Y denote the set of paths for which A and B both have odd area. Because A and B together form a square of area $4n^2$, which is even, $|X|+|Y|$ equals the total number of paths from $(0,0)$ to $(2n,2n)$, which is $\binom{4n}{2n}$.

Denoting a step to the right by R and a step upwards by U, every path from $(0,0)$ to $(2n,2n)$ can be described as a sequence of $4n$ symbols, $2n$ of which are R and $2n$ of which are U. We subdivide such a sequence into $2n$ pairs of consecutive steps that can be RR, UR, RU or UU. The number of possible paths for which neither UR nor RU occurs is $\binom{2n}{n}$, because out of $2n$ pairs that can be either RR or UU we have to choose n that will be RR. These $\binom{2n}{n}$ all belong to X; in fact, for these paths, A and B can be subdivided into 2×2-square, making their areas divisible by 4. Now consider the paths that contain at least one UR- or RU-pair. If in such a path we replace the first occurrence of a UR- or RU-pair by a pair of the other type (thus replacing UR by RU or vice versa), the areas of A and B each change by 1 and therefore become even if they were odd and odd if they were even. Because this modification is clearly reversible, we conclude that we can pair up all paths that contain at least one UR- or RU-pair into pairs of paths, one of which belongs to X and one of belongs to Y. This implies that $|X|-\binom{2n}{n}=|Y|$. It follows that

$$|X|=\frac{|X|+|Y|}{2}+\frac{|X|-|Y|}{2}=\frac{\binom{4n}{2n}}{2}+\frac{\binom{2n}{n}}{2}.$$

□

Solution 2

Define $Z_{m,n}$ to be the number of routes from $(0,0)$ to $(2m,2n)$ that divide the rectangle with vertices $(0,0)$, $(0,2n)$, $(2m,2n)$ and $(2m,0)$ into two parts of even area. We call such paths *good*. We claim that

$$2Z_{m,n}=\binom{2m+2n}{2m}+\binom{m+n}{m}$$

for all m,n, which for $m=n$ establishes the desired result. We prove this formula by induction on $m+n$, noting first that it clearly holds when either m or n is zero, because $Z_{0,n}=Z_{m,0}=1$ (there is only one path from $(0,0)$ to $(0,2n)$ or $(2m,0)$, which is good). Therefore, suppose that $m,n\geq 1$ and consider a good path from $(0,0)$ to $(2m,2n)$. This path passes through exactly one of $(2m,2n-2)$,

$(2m-1, 2n-1)$ and $(2m-2, 2n)$. If it passes through $(2m, 2n-2)$ then the subpath from $(0,0)$ to $(2m, 2n-2)$ must also be good; moreover, for each good path from $(0,0)$ to $(2m, 2n-2)$ there is exactly one corresponding path from $(0,0)$ to $(2m, 2n)$, and that path is automatically good because the new area that gets added is even. Thus, there are $Z_{m,n-1}$ good paths from $(0,0)$ to $(2m, 2n)$ that pass through $(2m, 2n-2)$. Similarly, there are $Z_{m-1,n}$ good paths from $(0,0)$ to $(2m, 2n)$ that pass through $(2m-2, 2n)$. Now notice that any path from $(0,0)$ to $(2m-1, 2n-1)$ (of which there are $\binom{2m+2n-2}{2m-1}$) can be extended in two ways to obtain a path from $(0,0)$ to $(2m, 2n)$; because the resulting areas for these paths differ by 1, exactly one of these paths is good. All in all, we obtain the recursion

$$Z_{m,n} = Z_{m,n-1} + Z_{m-1,n} + \binom{2m+2n-2}{2m-1}.$$

By the inductive hypothesis, we have

$$2Z_{m,n-1} = \binom{2m+2n-2}{2m} + \binom{m+n-1}{m}$$

and

$$2Z_{m-1,n} = \binom{2m+2n-2}{2m-2} + \binom{m+n-1}{m-1}.$$

Therefore, we obtain

$$\begin{aligned} 2Z_{m,n} &= 2Z_{m,n-1} + 2Z_{m-1,n} + 2\binom{2m+2n-2}{2m-1} \\ &= \binom{2m+2n-2}{2m} + \binom{m+n-1}{m} + \binom{2m+2n-2}{2m-2} + \binom{m+n-1}{m-1} \\ &\quad + 2\binom{2m+2n-2}{2m-1}. \end{aligned}$$

To simplify the expression on the right hand side, note that we can reorganize the terms

$$\binom{2m+2n-2}{2m} + \binom{2m+2n-2}{2m-2} + 2\binom{2m+2n-2}{2m-1}$$

as

$$\left(\binom{2m+2n-2}{2m} + \binom{2m+2n-2}{2m-1}\right) + \left(\binom{2m+2n-2}{2m-1} + \binom{2m+2n-2}{2m-2}\right),$$

which, using the addition rules for binomial coefficients, becomes

$$\binom{2m+2n-1}{2m} + \binom{2m+2n-1}{2m-1} = \binom{2m+2n}{2m}.$$

Similarly, we have

$$\binom{m+n-1}{m} + \binom{m+n-1}{m-1} = \binom{m+n}{m}.$$

Putting everything together, we obtain that

$$2Z_{m,n} = \binom{2m+2n}{2m} + \binom{m+n}{m},$$

which completes the induction. □

Remark

The result from Solution 2 can also be proved using a combinatorial argument like in Solution 1.

Solution 3

We start by proving the following lemma: for a

$$(2m - 1, 2n - 1)$$

-grid, there are equally many paths with a region of even area (called even paths), as there are with odd area (odd paths).

To prove this, we take a path and rotate it around the center of the grid. Then a path spanning a region with area x is mapped on one spanning an area $(2m - 1)(2n - 1) - x$. This gives a bijection between paths creating an even region, and creating an odd region.

Now, for every path from

$$(0, 0)$$

to $(2n, 2n)$, consider all the coordinates of the grid points it visits in order. There are $\binom{2n}{n}$ of them which never visit a point with odd coordinates (which we call an odd point). Notice that such paths are all even.

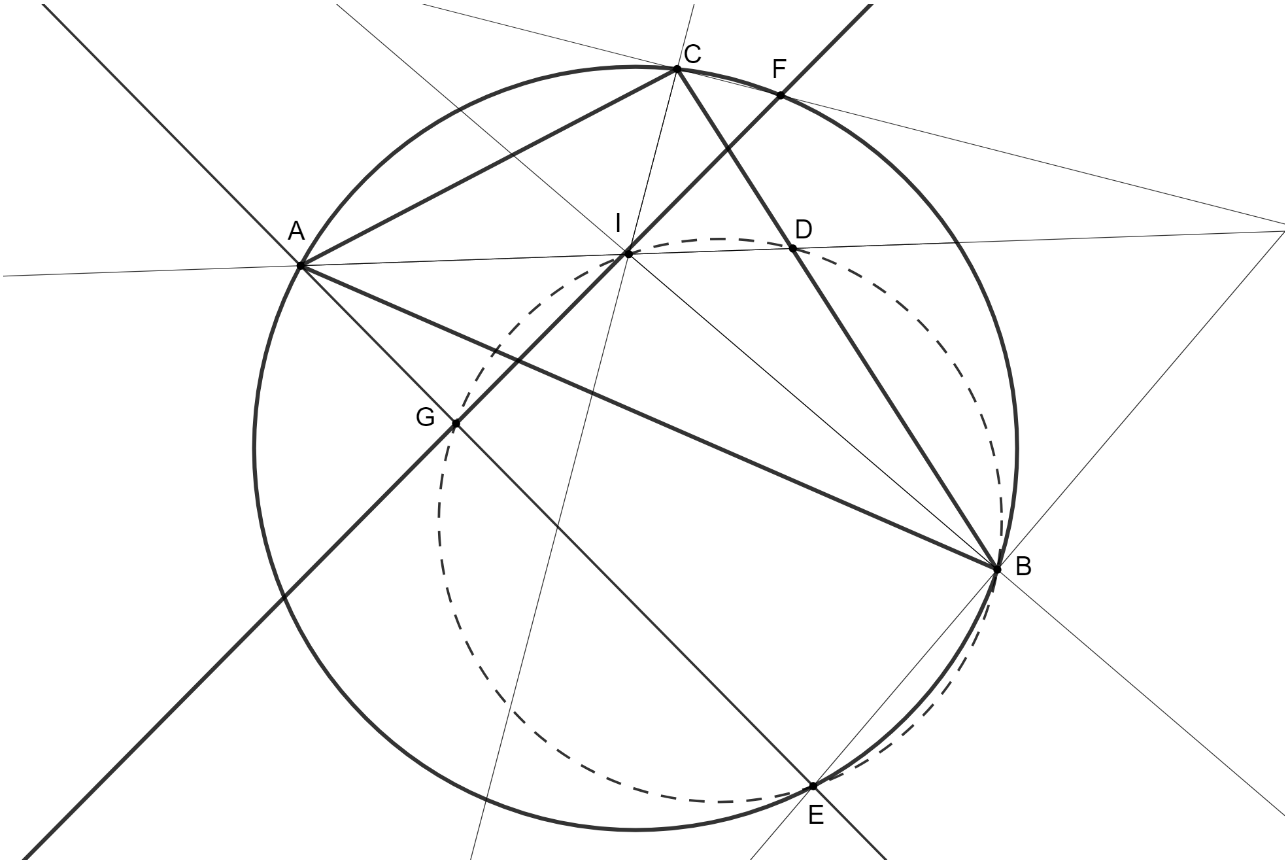
We now construction a bijection between the remaining even paths and the odd paths. For each odd point, there are equally many even as odd paths from $(0, 0)$ to that point. Define then a bijection ϕ between the sets of odd paths and even paths up to this point for each point. Notice that ϕ implicitly depends on the chosen odd point.

Now, for an arbitrary odd path P, consider the first odd point it passes through. Map P to another path by changing the path up to this odd point to the ϕ of the path up to this point. As ϕ maps between even and odd paths, the resulting path is an even path. By the definition of ϕ, and as each of the remaining paths goes to an odd point, this mapping defines a bijection.

We have hence found a bijection between the odd and even paths in the remaining $\binom{4n}{2n} - \binom{2n}{n}$ paths, which yields the required result like in solution 1.

Problem 3

Let ABC be a triangle with incentre I and circumcircle Ω such that $|AC| \neq |BC|$. The internal angle bisector of $\angle CAB$ intersects side $[BC]$ in D, and the external angle bisectors of $\angle ABC$ and $\angle BCA$ intersect Ω again in E and F, respectively. Let G be the intersection of lines AE and FI and let Γ be the circumcircle of triangle BDI. Show that E lies on Γ if and only if G lies on Γ.



Solution 1

We first notice the general fact that $EF \perp AI$. This can be proved using the following argument. Denote S for the intersection of EF and AI. Then $\angle BIS = (\angle IBA + \angle IAB) = \frac{1}{2}(\angle ABC + \angle BAC) = \frac{1}{2}(180° - \angle BCA) = \angle BCF = \angle BEF = \angle BES$. Thus B, I, S, E form a cyclic quadrilateral, from which it can be concluded that $EF \perp AI$.

We will now continue to prove the problem by doing both directions separately. Assume first that E lies on Γ. Then, as angle bisectors are perpendicular to one another, we have that $\angle IDE = \angle IBE = 90°$. Then, as $EF \perp AI$, it holds that D lies on EF. It can then be concluded that $\angle IDF = 90° = \angle ICF$ (again due to perpendicular bisectors), from which it can be concluded that I, D, C, F form a cyclic quadrilateral. We can now calculate that $\angle GID = 180° - \angle FID = 180° - \angle FCD = 180° - \angle FCB = \angle AEF = \angle GED$, where the second to last step follows from the fact that the arcs BF and AF have the same length, as F lies on the external bisector of $\angle ACB$. It now follows that I, D, E, G form a cyclic quadrilteral, thus G lies on Γ.

For the reverse implication, assume that G lies on Γ. We can then compute that $\angle FAD = \angle FAS = 90^\circ - \angle AFS = 90^\circ - \angle AFE = 90^\circ - \angle ABE = \angle ABI = \angle IBD = \angle IGD = \angle FGD$, from which we can conclude that A, G, D, F form a cyclic quadrilateral. Similarly to $\angle FAD = \angle IBD$, it holds that $\angle GAD = \angle ICD$. From this we can compute that $\angle IFD = \angle GFD = \angle GAD = \angle ICD$, thus I, D, C, F forms a cyclic quadrilateral. Hence, $\angle IDF = 180^\circ - \angle ICF = 90^\circ$, so again $D = S$. We then see that $\angle IBE = 90^\circ = \angle IDE$ from which we can conclude that I, D, E, B form a cyclic quadrilateral. It then follows that E lies on Γ.

Solution 2

The external angle bisectors BE and CF meet the internal bisector ID at the A-excentre J of triangle ABC. If one of $BEDI$ and $CFID$ is cyclic, then, as $BECF$ is cyclic, $|JD|\,|JI| = |JE|\,|JB| = |JC|\,|JF|$ by power of a point, and so the other is cyclic, too. This shows that

(1) $BEDI$ is cyclic if and only if $CFID$ is cyclic.

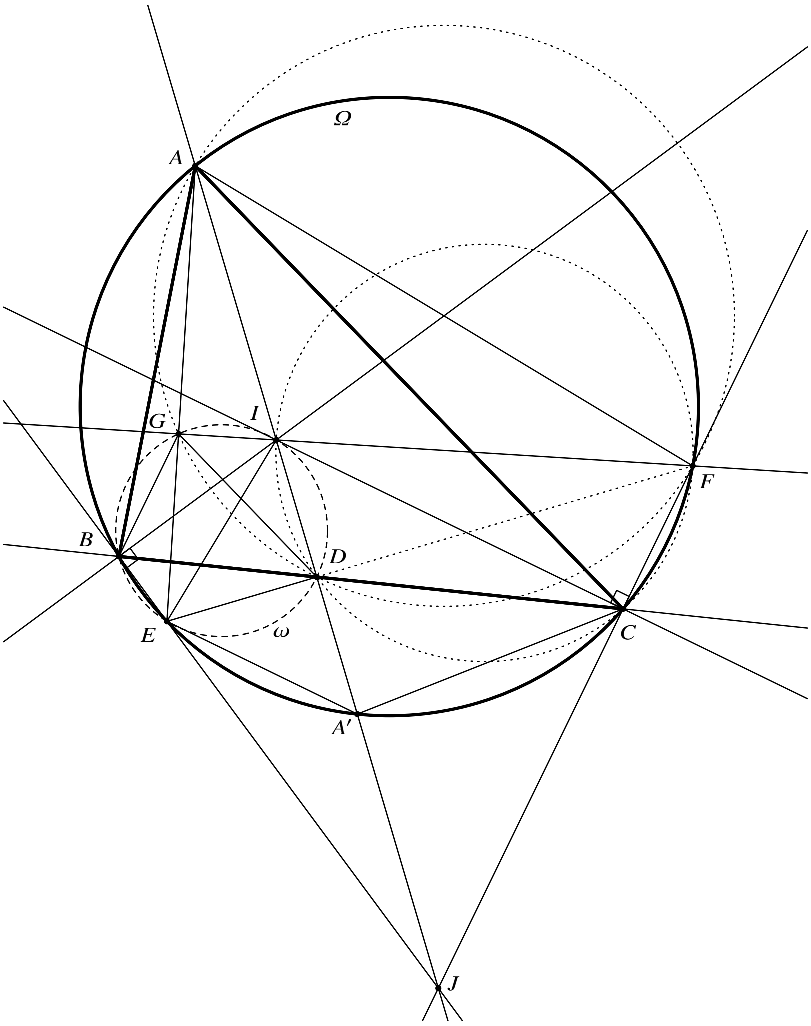
Next, $\angle GAD = \angle EAD = \angle BAD - \angle BAE = \angle A/2 - \left(180^\circ - \angle AEB - \angle EBA\right)$, with $\angle AEB = \angle ACB = \angle C$ and $\angle EBA = \angle EBI + \angle IBA = 90^\circ + \angle B/2$. Hence $\angle GAD = \angle A/2 + \angle B/2 + \angle C - 90^\circ = \angle C/2 = \angle ICD$. Thus, if one of $CFID$ and $AGDF$ is cyclic, then $\angle ICD = \angle IFD = \angle GFD = \angle GAD$, and the other is cyclic, too. We have thus established that

(2) $CFID$ is cyclic if and only if $AGDF$ is cylic.

Finally, let A' denote the second intersection of AI with Ω. Then $\angle DAF = \angle A'AF = 180^\circ - \angle FCA'$, with $\angle FCA' = \angle FCI + \angle ICB + \angle BCA' = 90^\circ + \angle C/2 + \angle BAA' = 90^\circ + \angle C/2 + \angle A/2$. It follows that $\angle DAF = 90^\circ - \angle A/2 - \angle C/2 = \angle B/2 = \angle DBI$. Hence, if one of $AGDF$ and $BDIG$ is cyclic, then $\angle DAF = \angle DGF = \angle DGI = \angle DBI$, and so the other is cyclic, too. Hence

(3) $AGDF$ is cyclic if and only if $BDIG$ is cylic.

These three equivalences prove that $BEDI$ is cyclic if and only if $BDIG$ is cyclic, i.e. E lies on ω if and only if G does. This completes the proof. □



Solution 3

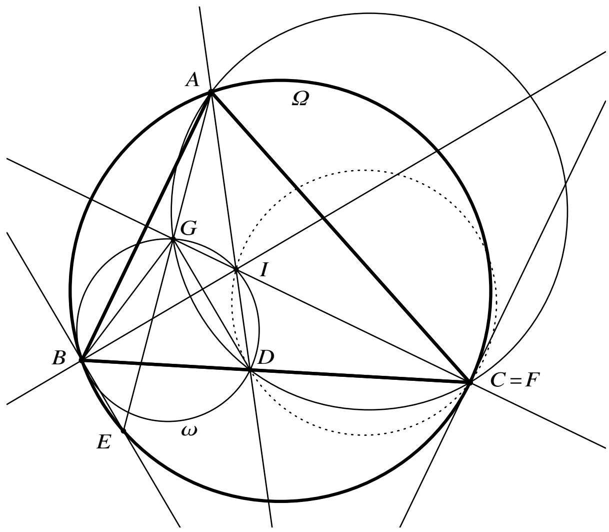
This proof only shows $E \in \omega \implies G \in \omega$. Note that this argument cannot be used straightforwardly to prove the converse implication.

If $BEDI$ is cyclic, let A' be the the second intersection of AI with Ω, so $\angle IA'E = \angle AA'E = 180° - \angle EBA$, with $\angle EBA = \angle EBI + \angle IBA = 90° + \angle B/2$, so $\angle IA'E = 90° - \angle B/2$. But $\angle EIA' = \angle EBD$ since $EBDI$ is cyclic, with $\angle EBD = \angle EBI - \angle DBI = 90° - \angle B/2$. Thus $\angle EA'I = \angle EIA'$, so $EA'I$ is isosceles. Moreover, since $EBDI$ is cyclic and $\angle EBI$ is a right angle, so is $\angle IDE$. It follows that D is the midpoint of $[AI]$. Next, the external bisectors BE and CF and the internal bisector ID meet at the A-excentre J of triangle ABC. By power of a point, since $BEDI$ and $BECF$ are cyclic, $|JD|\,|JI| = |JE|\,|JB| = |JC|\,|JF|$, so $CFID$ is cyclic, too, with $\angle FDI = \angle FCI = 90°$. We have thus shown that $\angle ADE = \angle FDA = 90°$, so D is the foot of the altitude from A in triangle AEF. Moreover, all of this shows that I is the reflection of A', which is the point at which this altitude meets the circumcircle Ω of AEF again, in the side $[EF]$. Hence I is the orthocentre of triangle AEF. By extension, FI is its altitude from F, and G is the foot of this altitude, so $\angle EGI = 90° = \angle IDE = \angle EBI$, and hence $BEDIG$ is cyclic. This shows that if E lies on

ω, then so does G. □

Remark

The equivalence $E \in \omega \iff G \in \omega$ breaks down if triangle ABC is not scalene. Indeed, if ABC is isosceles with $\angle A = \angle B$, then $F = C$ and G is the intersection of CI and AE. Angle chasing as above then shows that $\angle GAD = \angle C/2 = \angle GCD$, so $AGDC$ is cyclic. Then $\angle DGI = \angle DGC = \angle DAC = \angle A/2 = \angle B/2 = \angle DBI$, so $BDIG$ is cyclic, i.e. $G \in \omega$. Next, the A-excentre J of triangle ABC is the intersection of BE, ID, and the tangent to Ω at C, which, being the external bisector of $\angle C$, is perpendicular to CI. Hence, if $E \in \omega$, too, i.e. if $BEDI$ were cyclic, then by power of a point, $|JC|^2 = |JE|\,|JB| = |JD|\,|JI|$, so the circumcircle of CID would be tangent to JC at C, implying $\angle CDI = \angle ICJ = 90°$. This is not however the case, unless $\angle B = \angle C$ and hence ABC is equilateral. This shows that the condition in the problem statement that ABC is scalene cannot be dropped.



Problem 4

For each positive integer n, let $\mathrm{rad}(n)$ denote the product of the distinct prime factors of n. Show that there exist integers $a, b > 1$ such that $\gcd(a, b) = 1$ and

$$\mathrm{rad}(ab(a+b)) < \frac{a+b}{2024^{2024}}.$$

For example, $\mathrm{rad}(20) = \mathrm{rad}(2^2 \cdot 5) = 2 \cdot 5 = 10$ *and* $\mathrm{rad}(18) = \mathrm{rad}(2 \cdot 3^2) = 2 \cdot 3 = 6$.

Solution 1

We show that the pair (a, b) of the form $a = 2^{p(p-1)}, b = 3^{p(p-1)} - 2^{p(p-1)}$ for sufficiently larger prime number p satisfies the inequality. First, notice that $\gcd(a, b) = \gcd(a, a+b) = 1$ indeed. In addition, see that $\mathrm{rad}(a) = 2$, and $\mathrm{rad}(a+b) = 3$. Because of Euler-Fermat, as $\phi(p^2) = p(p-1)$, it can directly be seen that $p^2 \mid b$. In this case, $\mathrm{rad}(b) \leqslant \frac{b}{p}$. It then follows that, as rad is multiplicative for coprime numbers, that

$$\mathrm{rad}(ab(a+b)) = \mathrm{rad}(a)\mathrm{rad}(b)\mathrm{rad}(a+b) \leqslant 2 \cdot 3 \cdot \frac{b}{p} \leqslant \frac{6}{p}(a+b).$$

Then, by choosing p such that $\frac{6}{p} < \frac{1}{2024^{2024}}$, we found a and b satifying the inequality.

Solution 2

We show that the pair (a, b) of the form $a = 3^{2^k}, b = 5^{2^k} - 3^{2^k}$ for sufficiently large k satisifies the inequality. Again, we have that $\gcd(a, b) = \gcd(a, a+b) = 1$. Similarly to solution 1, $\mathrm{rad}(a(a+b)) = \mathrm{rad}(a)\mathrm{rad}(a+b) = 3 \cdot 5 = 15$. Then, we will show that $2^{k+1} \mid b$, from which it would follow that $\mathrm{rad}(b) \leqslant \frac{b}{2^k}$. From this, we then see that

$$\mathrm{rad}(ab(a+b)) = \mathrm{rad}(a(a+b))\mathrm{rad}(b) \leqslant \frac{15b}{2^k}.$$

Like in solution 1, this gives a pair (a, b) satisfying the inequality for sufficiently large k.

There are various ways to show that $2^{k+1} \mid 5^{2^k} - 3^{2^k}$, for example directly by applying the Lifting-The-Exponent Lemma. For a more elementary proof, we can apply induction on k. The statement is clearly true for $k = 0$, and if the statement holds for $k = n$, then for $k = n+1$ we see that

$$5^{2^k} - 3^{2^k} = 5^{2^{n+1}} - 3^{2^{n+1}} = (5^{2^n})^2 - (3^{2^n})^2 = (5^{2^n} - 3^{2^n})(5^{2^n} + 3^{2^n}).$$

From the induction hypothesis, the first factor has $n+1$ factors of 2. As the second factor is a sum of two odd numbers, the second term has at least one factor of 2. The product thus has at least $n+2 = k+1$ factors of 2, from which the statement follows by induction.

Solution 3

Choose $a = (4^x - 1)^2, b = 4^{x+1}$ and $a + b = (4^x + 1)^2$. That is, a, b and $a+b$ are squares, where b only contains the factor 2. Note that a and b are indeed coprime. Then $\mathrm{rad}(abc) = 2\mathrm{rad}(16^x - 1)$.

Choose then $x = 5^k$ such that $x > 2 \cdot 2024^{2024}$. By Lifting the exponent, we know that $5^{k+1} \mid 16^{5^k} - 1$. This implies that $2\mathrm{rad}(16^x - 1) \leqslant 2(16^x - 1)/5^k < (4^x + 1)^2/2024^{2024}$.

This solution also works with Euler-Fermat, by choosing $x = \phi(5^{k+1})$ with $5^k > 2024^{2024}$.