16th Benelux Mathematical Olympiad

Valkenswaard, 26th – 28th April 2024

Language: **English**

The problems are not ordered by estimated difficulty.

Problem 1. (a) Let $a_0, a_1, \ldots, a_{2024}$ be real numbers such that $|a_{i+1} - a_i| \leqslant 1$ for $i = 0, 1, \ldots, 2023$. Find the minimum possible value of

$$a_0a_1 + a_1a_2 + \cdots + a_{2023}a_{2024}.$$

(b) Does there exist a real number C such that

$$a_0a_1 - a_1a_2 + a_2a_3 - a_3a_4 + \cdots + a_{2022}a_{2023} - a_{2023}a_{2024} \geqslant C$$

for all real numbers $a_0, a_1, \ldots, a_{2024}$ such that $|a_{i+1} - a_i| \leqslant 1$ for $i = 0, 1, \ldots, 2023$?

Problem 2. Let n be a positive integer. In a coordinate grid, a *path* from $(0,0)$ to $(2n, 2n)$ consists of $4n$ consecutive unit steps $(1,0)$ or $(0,1)$. Prove that the number of paths that divide the square with vertices $(0,0)$, $(2n,0)$, $(2n,2n)$, $(0,2n)$ into two regions with even areas is

$$\frac{\binom{4n}{2n} + \binom{2n}{n}}{2}.$$

Problem 3. Let ABC be a triangle with incentre I and circumcircle Ω such that $|AC| \neq |BC|$. The internal angle bisector of $\angle CAB$ intersects side $[BC]$ in D, and the external angle bisectors of $\angle ABC$ and $\angle BCA$ intersect Ω again in E and F, respectively. Let G be the intersection of lines AE and FI and let Γ be the circumcircle of triangle BDI. Show that E lies on Γ if and only if G lies on Γ.

Problem 4. For each positive integer n, let $\operatorname{rad}(n)$ denote the product of the distinct prime factors of n. Show that there exist integers $a, b > 1$ such that $\gcd(a, b) = 1$ and

$$\operatorname{rad}(ab(a+b)) < \frac{a+b}{2024^{2024}}.$$

For example, $\operatorname{rad}(20) = \operatorname{rad}(2^2 \cdot 5) = 2 \cdot 5 = 10$ *and* $\operatorname{rad}(18) = \operatorname{rad}(2 \cdot 3^2) = 2 \cdot 3 = 6$.

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Time: 4 hours and 30 minutes.
Each problem is worth 7 points.