15th Benelux Mathematical Olympiad

Esch-sur-Alzette, 5th – 7th May 2023

Problems and Solutions

Problem Selection Committee
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BxMO 2023: Problems and Solutions

Problem 1

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(x-y)\big(f(x)+f(y)\big) \leqslant f\big(x^2-y^2\big) \qquad \text{for all } x, y \in \mathbb{R}.$$

Solution

Clearly, $f(x) = cx$ is a solution for each $c \in \mathbb{R}$ since $(x-y)(cx+cy) = c\big(x^2-y^2\big)$. To show that there are no other solutions, we observe that

(1) $x = y$: $\quad 0 \leqslant f(0)$;
$x = 1, y = 0$: $\quad f(0) + f(1) \leqslant f(1) \Rightarrow f(0) \leqslant 0$, whence $f(0) = 0$;

(2) $y = -x$: $\quad 2x\big(f(x) + f(-x)\big) \leqslant f(0) = 0$;
$x \to -x$: $\quad -2x\big(f(-x) + f(x)\big) \leqslant 0 \Rightarrow 2x\big(f(x) + f(-x)\big) \geqslant 0$;
thus $2x\big(f(x) + f(-x)\big) = 0$ for all x, so $f(-x) = -f(x)$ for all $x \neq 0$, and hence for all x, since $f(0) = 0$;

(3) $x \leftrightarrow y$: $\quad (y-x)\big(f(y) + f(x)\big) \leqslant f\big(y^2 - x^2\big) = -f\big(x^2 - y^2\big) \Rightarrow (x-y)\big(f(x) + f(y)\big) \geqslant f\big(x^2 - y^2\big)$;
which is the given inequality with the inequality sign reversed, so $(x-y)\big(f(x) + f(y)\big) = f\big(x^2 - y^2\big)$ must hold for all $x, y \in \mathbb{R}$;

(4) $y \leftrightarrow -y$: $\quad (x-y)\big(f(x)+f(y)\big) = f\big(x^2-y^2\big) = f\big(x^2-(-y)^2\big) = (x+y)\big(f(x)+f(-y)\big) = (x+y)\big(f(x)-f(y)\big)$;
expanding yields $f(x)y = f(y)x$ for all $x, y \in \mathbb{R}$.

Taking $y = 1$ in the last result, $f(x) = f(1)x$, i.e. $f(x) = cx$, where $c = f(1)$, for all $x \in \mathbb{R}$. Since we have shown above that, conversely, all such functions are solutions, this completes the proof. □

Alternative solution. A slight variation of this argument proves that $(x-y)\big(f(x) + f(y)\big) = f\big(x^2 - y^2\big)$ must hold for all $x, y \in \mathbb{R}$ as above, and then reaches $f(x) = cx$ as follows:

(4) $y = \pm 1$: $\quad (x \mp 1)\big(f(x) \pm f(1)\big) = f\big(x^2 - 1\big)$ using $f(-1) = -f(1)$ from (2);
hence $(x-1)\big(f(x) + f(1)\big) = (x+1)\big(f(x) - f(1)\big) \Rightarrow f(x) = f(1)x = cx$, where $c = f(1)$, on expanding.

Problem 2

Determine all integers $k \geqslant 1$ with the following property: given k different colours, if each integer is coloured in one of these k colours, then there must exist integers $a_1 < a_2 < \cdots < a_{2023}$ of the same colour such that the differences $a_2 - a_1, a_3 - a_2, \ldots, a_{2023} - a_{2022}$ are all powers of 2.

Solution

We claim that only $k = 1$ and $k = 2$ satisfy the required property. First, if $k \geqslant 3$, we colour each integer with its residue class modulo 3, so that, whenever two integers have the same colour, their difference is divisible by 3, so is not a power of 2. This shows that no $k \geqslant 3$ has the required property.

In the case $k = 1$, the sequence defined by $a_n = 2n$ for $n = 1, 2, \ldots, 2023$ clearly has the required property. In the case $k = 2$, we call the colours "red" and "blue", and construct, for each $n \geqslant 1$ and by induction, integers $a_1 < a_2 < \cdots < a_n$ of the same colour such that $a_{m+1} - a_m$ is a power of 2 for $m = 1, 2, \ldots, n-1$. The statement is trivial for $n = 1$. For $n > 1$, let $a_1 < a_2 < \cdots < a_n$ be red integers (without loss of generality) having the desired property. Consider the $n + 1$ integers $b_i = a_n + 2^i$, for $i = 1, 2, \ldots, n + 1$. If one of these, say b_j, is red, then, as $b_j - a_n = 2^j$, the $n + 1$ red integers $a_1 < a_2 < \cdots < a_n < b_j$ have the desired property. Otherwise, $b_1, b_2, \ldots, b_{n+1}$ are all blue, and $b_{i+1} - b_i = \left(a_n + 2^{i+1}\right) - \left(a_n + 2^i\right) = 2^i$ for $i = 1, 2, \ldots, n$, so the $n+1$ blue integers $b_1 < b_2 < \cdots < b_{n+1}$ have the desired property. This completes the inductive step and hence the proof. $\square$

Problem 3

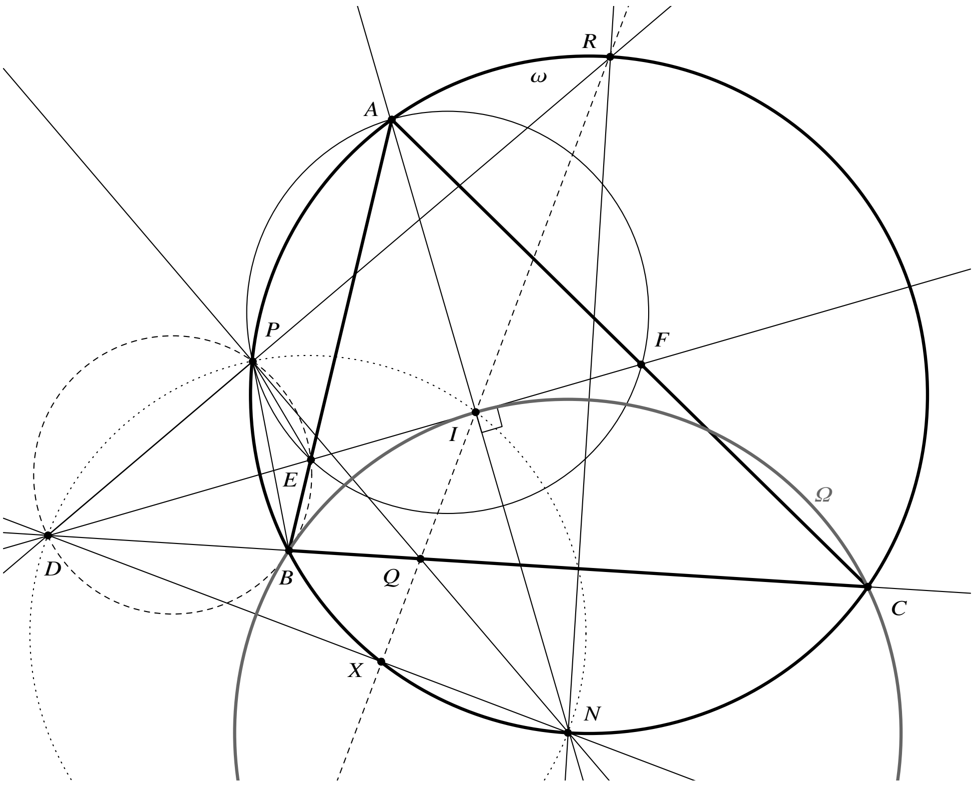
Let ABC be a triangle with incentre I and circumcircle ω. Let N denote the second point of intersection of line AI and ω. The line through I perpendicular to AI intersects line BC, segment $[AB]$, and segment $[AC]$ at the points D, E, and F, respectively. The circumcircle of triangle AEF meets ω again at P, and lines PN and BC intersect at Q. Prove that lines IQ and DN intersect on ω.

Solution 1

By construction, $APEF$ and $APBC$ are cyclic, and so

$$\begin{aligned}\angle BDE = \angle CDF = \angle AFD - \angle FCD = \angle AFE - \angle ACB = \left(180^\circ - \angle EPA\right) - \left(180^\circ - \angle BPA\right)\\ = \angle BPA - \angle EPA = \angle BPE.\end{aligned}$$

Hence $DBEP$ is cyclic, too. It follows that $\angle IDP = \angle EDP = \angle EBP = \angle ABP = \angle ANP = \angle INP$ since $APBN$ is cyclic, and so $PDNI$ is also cyclic. In particular, $\angle DPN = \angle DIN = 90^\circ$. Let R denote the second intersection of DP and ω, so $NQ \perp DR$. Then $\angle NPR = 90^\circ$, so RN is a diameter of ω. It is well-known that N is the midpoint of the arc $\widehat{BC}$ not containing A, whence $RN \perp BC$. Thus DQ and NQ are altitudes of triangle RDN, and so Q is its orthocentre. This implies that $RQ \perp DN$, whence, since RN is a diameter of ω, the intersection X of RQ and DN lies on ω.



It is also well-known that N is the centre of the circumcircle Ω of triangle BCI. Since $DI \perp IN$ by construction, DI is tangent to Ω at I. As D lies on the radical axis BC of ω and Ω, it follows that $|DI|^2 = |DB|\,|DC| = |DX|\,|DN|$. Hence triangles DNI and DIX are similar; in particular, $\angle DXI = \angle DIN = 90^\circ$. All of this shows that R, I, Q, X lie on a line perpendicular to DN that intersects DN at $X \in \omega$. This completes the proof. □

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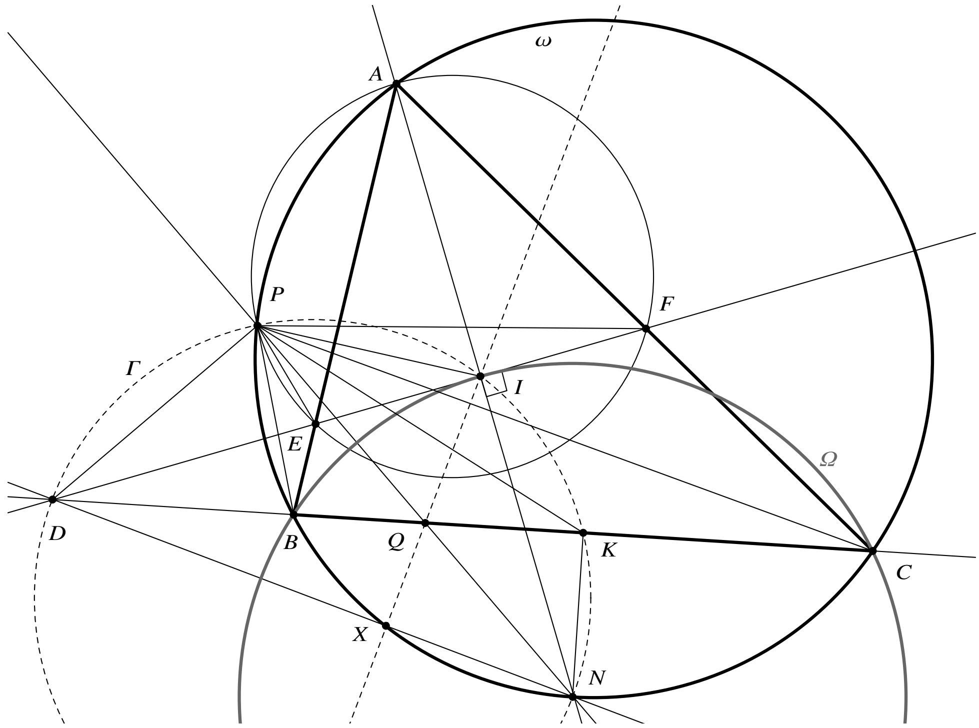
Solution 2

Let K be the midpoint of segment $[BC]$. It is well-known that N is the midpoint of the small arc $\widehat{BC}$ of ω, so $BC \perp KN$. In particular, $\angle DKN = 90°$. But $\angle DIN = 90°$ by construction, so $DIKN$ is cyclic, with circumcircle Γ.

Moreover, $\angle PEF = 180° - \angle PAF = 180° - \angle PAB = \angle PBC$ and $\angle PFE = \angle PAE = \angle PAB = \angle PCB$ since $AFEP$ and $ACBP$ are cyclic, so triangles PEF and PBC are similar. Now, by construction, I is the midpoint of segment $[EF]$, so, K being the midpoint of $[BC]$, triangles PIF and PKC are similar, too. It follows that $\angle PID = 180° - \angle PIF = 180° - \angle PKC = \angle PKD$, whence P lies on Γ.



Let Ω be the circumcircle of triangle BCI. By construction, Q lies on the radical axes PN of ω, Γ and BC of ω, Ω, so is the radical centre of ω, Γ, Ω. In particular, IQ is the radical axis of Γ, Ω, so is perpendicular to the line joining the centres of Γ, Ω. Now it is well-known that N is the centre of Ω, and, since $\angle DIN = 90°$, the centre of Γ is the midpoint of segment $[DN]$. This shows that $IQ \perp DN$.

Finally, let DN meet ω again at X. Since $DI \perp IN$ by construction and N is the centre of Ω, DI is tangent to Ω at I. As D lies on the radical axis BC of ω, Ω, it follows that $|DI|^2 = |DB|\,|DC| = |DX|\,|DN|$. Hence triangles DNI and DIX are similar; in particular, $\angle DXI = \angle DIN = 90°$, i.e. $IX \perp DN$. Since $IQ \perp DN$, it follows that X is the intersection of IQ and DN. Since X lies on ω by construction, this completes the proof. □

Solution 3

Since $APEF$ and $APBC$ are cyclic,

$$\begin{aligned}\angle CPF &= \angle BPA - \angle BPC - \angle FPA = (180° - \angle BCA) - \angle BAC - \angle FEA \\ &= (180° - \angle BCA - \angle BAC) - \angle BED = \angle CBA - \angle BED = \angle CBE - \angle BED = \angle BDE = \angle CDF,\end{aligned}$$

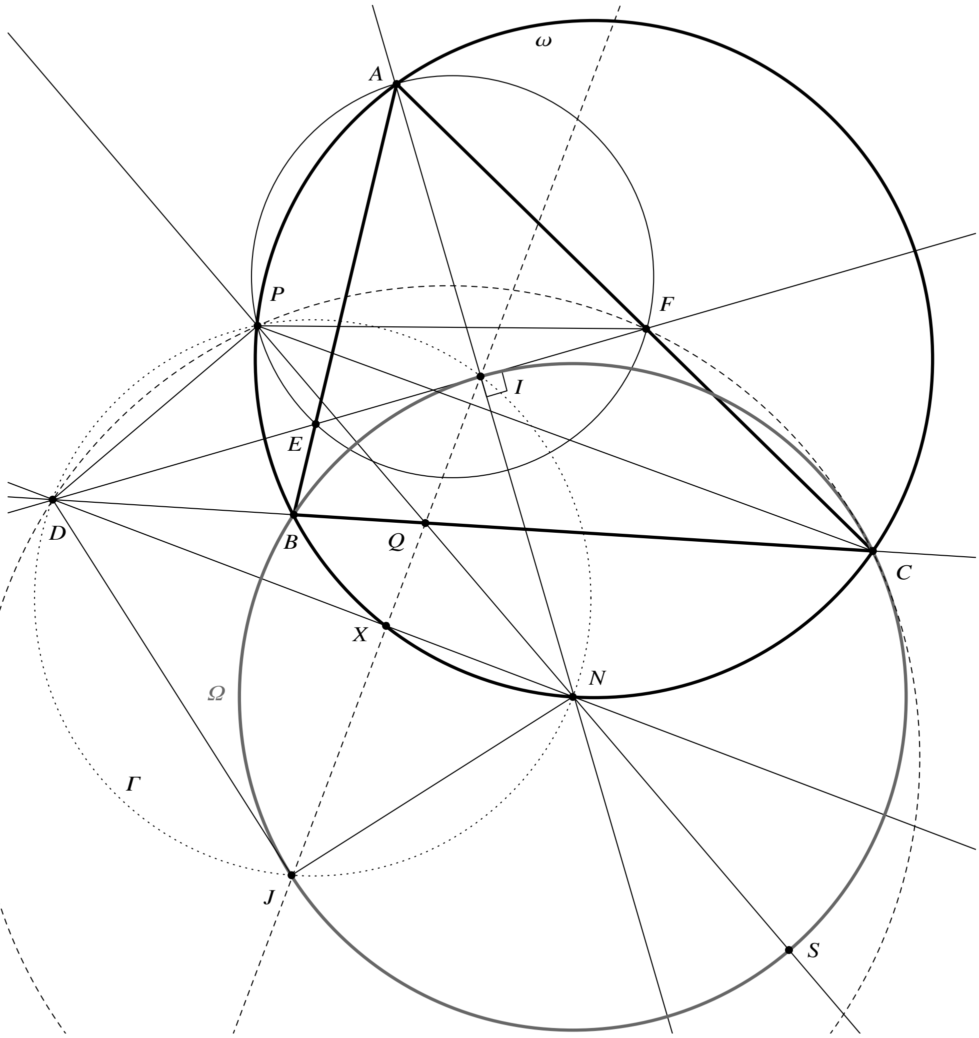
so $DPFC$ is cyclic, too. Thence $\angle CPD = \angle CFD = 180° - \angle IFA = 90° + \angle IAF = 90° + \angle CAN = 90° + \angle CPN$. Hence $\angle DPN = \angle CPD - \angle CPN = 90°$. Since $\angle DIN = 90°$ by construction, it follows that $DPIN$ is cyclic, with circumcircle Γ. Let J be the second intersection of line IQ and Γ. Moreover, it is well-known that N is the centre of the circumcircle Ω of BIC. In particular, $|NI| = |NB|$, and so, since $NIPJ$ and $NBPC$ are cyclic,

$$\frac{|JQ|}{|JP|} = \frac{|NQ|}{|NI|} = \frac{|NQ|}{|NB|} = \frac{|CQ|}{|CP|}. \tag{1}$$

Let S now be the point of intersection of PN and Ω such that P, N, S lie on line PN in this order. By construction, $\angle QPC = \angle NPC = \angle NAC = \angle BAN = \angle BCN = \angle QCN$, so triangles CQN and PCN are similar, whence

$$\frac{|CQ|}{|CP|} = \frac{|NC|}{|NP|} = \frac{|NQ|}{|NC|} = \frac{|NC| + |NQ|}{|NC| + |NP|} = \frac{|NS| + |NQ|}{|NS| + |NP|} = \frac{|SQ|}{|SP|}. \tag{2}$$

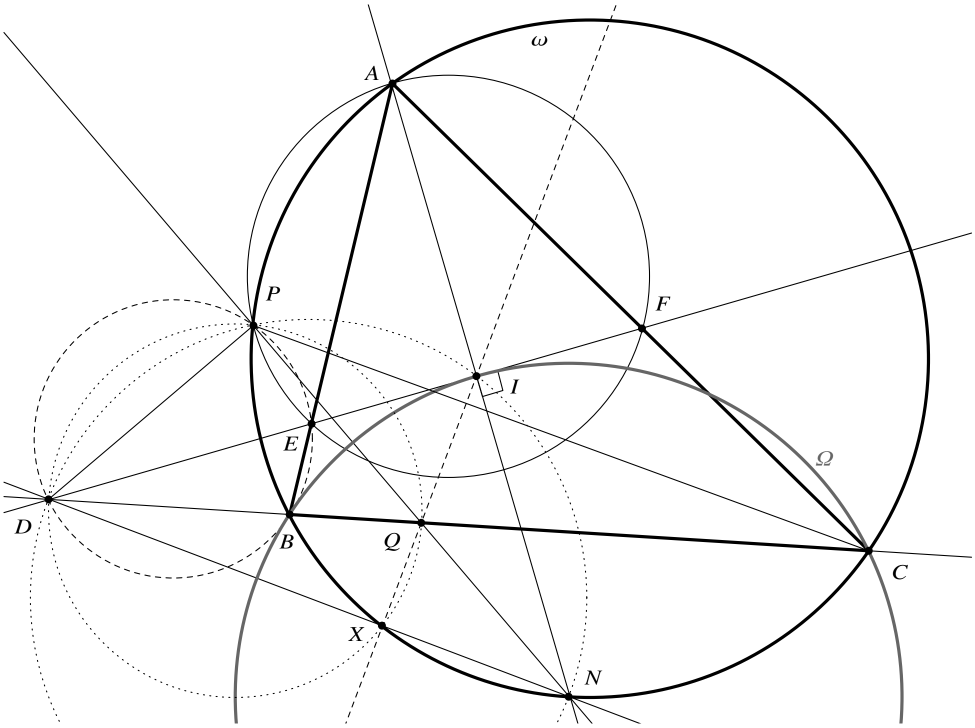
Combining (1) and (2) shows that C, J, S lie on a circle of Apollonius, the centre of which lies on the line through P, Q, N, S, so, since $|NC| = |NS|$ by construction, is N. In other words, J lies on Ω.



In particular, $|NI| = |NJ|$. Now, by construction, $\angle DIN = \angle DJN = 90°$, so the right-angled triangles DIN and DJN are congruent, whence $DINJ$ is a kite. In particular, $IJ \perp DN$. Since Q lies on IJ by definition, this shows that $IQ \perp DN$. We can now conclude as in **Solution 2**. □

Solution 4

By construction, P is the Miquel point of quadrilateral $BCFE$ (and the resulting complete quadrilateral with points A and D added) because it is the intersection of ω and the circumcircle of triangle AEF. In particular, $DBEP$ is cyclic. It follows that $\angle IDP = \angle EDP = \angle EBP = \angle ABP = \angle ANP = \angle INP$ since $APBN$ is cyclic, and so $PDNI$ is also cyclic.



Next, let X be the intersection of DN and ω and let AN meet BC at Y. Then $\angle NAC = \angle A/2 = \angle NCB$, so $\angle BYA = \angle C + \angle NAC = \angle C + \angle NCB = \angle NCA$ and hence

$$\begin{aligned}\angle DQP &= \angle NQY = \angle QYA - \angle QNY = \angle BYA - \angle PNA \\ &= \angle NCA - \angle PCA = \angle PCN = 180^\circ - \angle NXP = \angle DXP.\end{aligned}$$

This implies that $DXQP$ is cyclic. In particular, $QX \perp DN$. It now suffices to show that $IX \perp DN$, which we do in the same way as in **Solution 2**. □

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Problem 4

A positive integer n is *friendly* if every pair of neighbouring digits of n, written in base 10, differs by exactly 1. *For example, 6787 is friendly, but 211 and 901 are not.*

Find all odd natural numbers m for which there exists a friendly integer divisible by $64m$.

Solution

Any friendly number divisible by 64 is divisible by 4, and hence the number formed by its last two digits is a multiple of 4, so ends in 00, 04, 08, ..., or 96. A friendly number divisible by 4 must therefore end in 12, 32, 56, or 76, so cannot be divisible by 5. In particular, if $5 \mid m$, then there is no friendly integer divisible by $64m$.

We claim that conversely, if m is odd and $5 \nmid m$, then there exists a friendly integer divisible by $64m$. First, we notice that $343232 = 64 \cdot 5363$ is a friendly number divisible by 64, and hence so is

$$N_k = 343232343232 \cdots 343232 = 343232 \cdot \left(1 + 10^6 + \cdots + 10^{6k}\right) \quad \text{for } k = 0, 1, 2, \ldots.$$

Now the sequence $N_0, N_1, N_2, \ldots$ eventually repeats modulo m, i.e. there exist positive integers $k < \ell$ such that $N_\ell \equiv N_k \pmod{m}$. Hence $m \mid N_\ell - N_k = 10^{6(k+1)} N_{\ell-k-1}$. Since m is odd and $5 \nmid m$, $(10, m) = 1$, so $m \mid N_{\ell-k-1}$. By construction, $64 \mid N_{\ell-k-1}$. Thus, as m is odd and hence $(64, m) = 1$, we conclude that $64m \mid N_{\ell-k-1}$. This completes the proof. □

The solution divides into two parts: (1) showing that, if $5 \mid m$, then there is no friendly integer divisible by $64m$; (2) showing that, if $5 \nmid m$, then there is a friendly integer divisible by $64m$.

Alternative solution for part (1). If $5 \mid m$, then $20 \mid 64m$. The last two digits of a multiple of 20 are 00, 20, 40, 60, or 80, so this number is not friendly. Thus, if m is odd and $5 \mid m$, then there is no friendly integer divisible by $64m$.

Alternative solution for part (2). Notice that $N_k = 343232 \cdot \left(10^{6(k+1)} - 1\right) / \left(10^6 - 1\right)$. Let $M = m\left(10^6 - 1\right)$. Since $5 \nmid m$ and m is odd, $(10, M) = 1$, so, taking $k = \varphi(M) - 1$, we get $10^{6(k+1)} = 10^{6\varphi(M)} \equiv 1 \pmod{M}$ by the Euler–Fermat theorem, i.e. $m \mid \left(10^{6(k+1)} - 1\right) / \left(10^6 - 1\right)$, and hence $m \mid N_k$.

Alternative constructions of the integers N_k for part (2). Direct calculation shows that friendly integers divisible by 64 end in 343232, 543232, 123456, or 323456, so the numbers N_k defined in the solution of part (2) above may be replaced by, for instance,

$$34543232 \cdot \left(1 + 10^8 + \cdots + 10^{8k}\right),\ 5432123456 \cdot \left(1 + 10^{10} + \cdots + 10^{10k}\right),\ 54323456 \cdot \left(1 + 10^8 + \cdots + 10^{8k}\right).$$

Remark. Interestingly, friendly numbers cannot be divisible by arbitrarily high powers of 2. Direct calculation shows that the 60-digit friendly integer 101232121234323456543434343210121212323434343234565656543232 is divisible by 2^{60}, but that there is no friendly integer divisible by 2^{61}. The problem selection committee is not aware of a proof of this fact that eschews direct calculation.