15th Benelux Mathematical Olympiad

Luxembourg, 5th – 7th May 2023

Language: **English**

The problems are not ordered by estimated difficulty.

Problem 1. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(x-y)\big(f(x)+f(y)\big) \leqslant f\big(x^2-y^2\big) \qquad \text{for all } x, y \in \mathbb{R}.$$

Problem 2. Determine all integers $k \geqslant 1$ with the following property: given k different colours, if each integer is coloured in one of these k colours, then there must exist integers $a_1 < a_2 < \cdots < a_{2023}$ of the same colour such that the differences $a_2 - a_1, a_3 - a_2, \ldots, a_{2023} - a_{2022}$ are all powers of 2.

Problem 3. Let ABC be a triangle with incentre I and circumcircle ω. Let N denote the second point of intersection of line AI and ω. The line through I perpendicular to AI intersects line BC, segment $[AB]$, and segment $[AC]$ at the points D, E, and F, respectively. The circumcircle of triangle AEF meets ω again at P, and lines PN and BC intersect at Q. Prove that lines IQ and DN intersect on ω.

Problem 4. A positive integer n is *friendly* if the difference of each pair of neighbouring digits of n, written in base 10, is exactly 1. *For example, 6787 is friendly, but 211 and 901 are not.*

Find all odd natural numbers m for which there exists a friendly integer divisible by $64m$.

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Time: 4 hours and 30 minutes.
Each problem is worth 7 points.