14th Benelux Mathematical Olympiad

Leuven, 29th April – 1st May 2022

Problems and Solutions

Problem Selection Committee
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Problem 1

Let $n \geqslant 0$ be an integer, and let $a_0, a_1, \ldots, a_n$ be real numbers. Show that there exists $k \in \{0, 1, \ldots, n\}$ such that

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \leqslant a_0 + a_1 + \cdots + a_k$$

for all real numbers $x \in [0, 1]$.

Solution 1

The case $n = 0$ is trivial; for $n > 0$, the proof goes by induction on n. We need to make one preliminary observation:

Claim. For all reals a, b, $a + bx \leqslant \max\{a, a + b\}$ for all $x \in [0, 1]$.

Proof. If $b \leqslant 0$, then $a + bx \leqslant a$ for all $x \in [0, 1]$; otherwise, if $b > 0$, $a + bx \leqslant a + b$ for all $x \in [0, 1]$. This proves our claim. □

This disposes of the base case $n = 1$ of the induction: $a_0 + a_1x \leqslant \max\{a_0, a_0 + a_1\}$ for all $x \in [0, 1]$. For $n \geqslant 2$, we note that, for all $x \in [0, 1]$,

$$\begin{aligned} a_0 + a_1x + \cdots + a_nx^n &= a_0 + x\bigl(a_1 + a_2x + \cdots + a_nx^{n-1}\bigr) \\ &\leqslant a_0 + x(a_1 + a_2 + \cdots + a_k) \leqslant \max\{a_0, a_0 + (a_1 + \cdots + a_k)\}, \end{aligned}$$

for some $k \in \{1, 2, \ldots, n\}$ by the inductive hypothesis and our earlier claim. This completes the proof by induction. □

Solution 2

Define $s_i = a_0 + a_1 + \cdots + a_i$ for $i \in \{0, 1, \ldots, n\}$. Thus $a_0 = s_0$ and $a_i = s_i - s_{i-1}$ for all $i \in \{1, 2, \ldots, n\}$. Hence

$$\begin{aligned} a_0 + a_1x + a_2x^2 + \cdots + a_nx^n &= s_0 + (s_1 - s_0)x + (s_2 - s_1)x^2 + \ldots + (s_n - s_{n-1})x^n \\ &= s_0(1 - x) + s_1(x - x^2) + \ldots + s_{n-1}(x^{n-1} - x^n) + s_nx^n. \end{aligned}$$

Now choose $k \in \{0, 1, \ldots, n\}$ such that $s_k = \max\{s_0, s_1, \ldots, s_n\}$. Using the inequality $x^{i-1} - x^i \geqslant 0$, valid for all $i \in \{1, 2, \ldots, n\}$ and all $x \in [0, 1]$, in the right-hand side above, it follows that

$$\begin{aligned} a_0 + a_1x + a_2x^2 + \cdots + a_nx^n &\leqslant s_k(1 - x) + s_k(x - x^2) + \cdots + s_k(x^{n-1} - x^n) + s_kx^n \\ &= s_k\left[(1 - x) + (x - x^2) + \cdots + (x^{n-1} - x^n) + x^n\right] \\ &= s_k = a_0 + a_1 + \cdots + a_k. \end{aligned}$$

This completes the proof. □

Solution 3

The proof proceeds by induction on n. The base case $n = 0$ is trivial. For $n \geqslant 1$, since $x \in [0, 1]$, we have $x^n \leqslant x^{n-1}$. Thus, if $a_n \geqslant 0$, then $a_n x^n \leqslant a_n x^{n-1}$, while, if $a_n < 0$, then $a_n x^n < 0$ trivially. This shows that $a_n x^n \leqslant \max\{0, a_n x^{n-1}\}$, whence

$$a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n \leqslant a_0 + a_1 x + \cdots + \max\{a_{n-1}, a_{n-1} + a_n\} x^{n-1}.$$

By the inductive hypothesis, the polynomial of degree $n - 1$ on the right-hand side is bounded above by $a_0 + \cdots + a_k$ for some $k \in \{0, 1, \ldots, n-2\}$ or $a_0 + \cdots + a_{n-2} + \max\{a_{n-1}, a_{n-1} + a_n\}$. But the latter is equal to one of $a_0 + a_1 + \cdots + a_{n-1}$ or $a_0 + a_1 + \cdots + a_n$; both are of the desired form, $a_0 + a_1 + \cdots + a_k$ for some $k \in \{n-1, n\}$. This completes the proof by induction. $\square$

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Problem 2

Let n be a positive integer. There are n ants walking along a line at constant nonzero speeds. Different ants need not walk at the same speed or walk in the same direction. Whenever two or more ants collide, all the ants involved in this collision instantly change directions. (Different ants need not be moving in opposite directions when they collide, since a faster ant may catch up with a slower one that is moving in the same direction.) The ants keep walking indefinitely.
Assuming that the total number of collisions is finite, determine the largest possible number of collisions in terms of n.

Solution 1

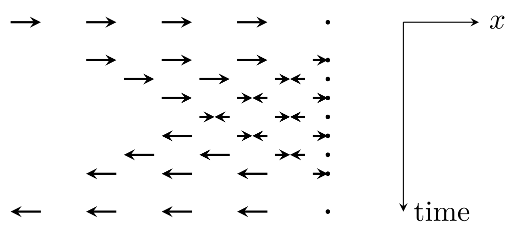
The order of the ants along the line does not change; denote by $v_1, v_2, \ldots, v_n$ the respective speeds of ants $1, 2, \ldots, n$ in this order. If $v_{i-1} < v_i > v_{i+1}$ for some $i \in \{2, \ldots, n-1\}$, then, at each stage, ant i can catch up with ants $i-1$ or $i+1$ irrespective of the latters' directions of motion, so the number of collisions is infinite. Hence, if the number of collisions is finite, then, up to switching the direction definining the order of the ants, (i) $v_1 \geqslant \cdots \geqslant v_n$ or (ii) $v_1 \geqslant \cdots \geqslant v_{k-1} > v_k \leqslant \cdots \leqslant v_n$ for some $k \in \{2, \ldots, n-1\}$. We need the following observation:

Claim. If $v_1 \geqslant \cdots \geqslant v_m$, then ants $m-1$ and m collide at most $m-1$ times.

Proof. The proof goes by induction on m, the case $m = 1$ being trivial. Since $v_{m-1} \geqslant v_m$, ants $m-1$ and m can only collide if the former is moving towards the latter. Hence, between successive collisions with ant m, ant $m-1$ must reverse direction by colliding with ant $m-2$. Since ants $m-1$ and $m-2$ collide at most $(m-1)-1 = m-2$ times by the inductive hypothesis, ants m and $m-1$ collide at most $(m-2)+1 = m-1$ times. □

Hence, in case (i), there are at most $0+1+\cdots+(n-1) = n(n-1)/2$ collisions. In case (ii), applying the claim to ants $1, 2, \ldots, k$ and also to ants $n, n-1, \ldots, k$ by switching their order, the number of collisions is at most $k(k-1)/2 + (n-k+1)(n-k)/2 = n(n-1)/2 - (k-1)(n-k) < n(n-1)/2$.

Now take a coordinate x along the line, and put ants at $x = 1, 2, \ldots, n$ with positive initial velocities and speeds $v_1 = \cdots = v_{n-1} = 1, v_n = \varepsilon$, for some ε. For $\varepsilon = 0$, collisions occur according to the pattern shown below for $n = 5$, which clearly extends to all values of n in such a way that ants m and $m+1$ collide exactly m times for $m = 1, 2, \ldots, n-1$. This yield $1+2+\cdots+(n-1) = n(n-1)/2$ collisions in total. For all sufficiently small $\varepsilon > 0$, the number of collisions remains equal to $n(n-1)/2$.



This shows that the upper bound obtained above can be attained. If the number of collisions is finite, the largest possible number of collisions is therefore indeed $n(n-1)/2$. □

Solution 2

We show that there are at most $n(n-1)/2$ collisions if the number of collisions is finite as in **Solution 1**.

To show that the upper bound of $n(n-1)/2$ collisions can be attained, we construct, inductively, an example of n ants colliding $n(n-1)/2$ times, the speeds of the ants decrease from left to right, and after all collisions all ants move towards the left, with the possible exception of the rightmost ant. In every case, we will label the ants $1, 2, \dots, n$ from left to right. For $n = 1$ this is trivial. For $n \geqslant 2$, we use the construction for $n-1$ ants (now labelled $2, 3, \dots, n$). We add ant 1 on the left, moving towards the right, faster than all other ants (so that the speeds of the ants still decrease from left to right), and in such a way that its first collision (with ant 2) happens after all $(n-1)(n-2)/2$ collisions of the other $n-1$ ants. Now the following events happen (in this order) for $i = 1, 2, \dots, n-2$: ants i and $i+1$ collide, after which ant i moves to the left and ant $i+1$ moves to the right. These collisions do happen because the speeds of the ants decrease from left to right. Then ants $n-1$ and n also collide, resulting in ant $n-1$ moving to the left. This shows that there are (at least) $(n-1)(n-2)/2 + (n-1) = n(n-1)/2$ collisions. There are in fact no more collisions since the speeds of the ants decrease from left to right; alternatively, this follows from the upper bound proved previously. Since all ants except ant n are moving towards the left after the collisions, this completes the inductive construction. □

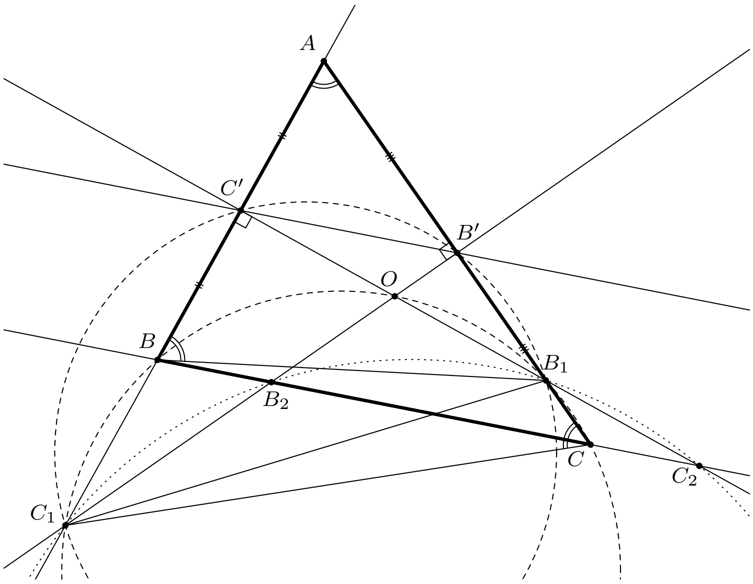
Problem 3

Let ABC be a scalene acute triangle. Let B_1 be the point on ray $[AC$ such that $|AB_1| = |BB_1|$. Let C_1 be the point on ray $[AB$ such that $|AC_1| = |CC_1|$. Let B_2 and C_2 be the points on line BC such that $|AB_2| = |CB_2|$ and $|BC_2| = |AC_2|$. Prove that B_1, C_1, B_2, C_2 are concyclic.

Solution 1

By construction, lines B_1C_2 and B_2C_1 bisect segments $[AB]$ and $[AC]$, respectively, so their intersection O is the circumcentre of ABC. Hence $\angle BOC = 2\angle A$ and $\angle CBO = \angle OCB = 90^\circ - \angle A$. Now, by construction, $\angle OC_1B = 90^\circ - \angle A = \angle OCB$, so BC_1CO is cyclic. Similarly, BCB_1O is cyclic by construction because $\angle OB_1C = 180^\circ - \angle AB_1O = 90^\circ + \angle A = 180^\circ - \angle CBO$. In particular, BC_1CB_1 is cyclic, too.

Now $\angle B_1C_1B_2 = \angle B_1C_1B - \angle B_2C_1B$ and $\angle B_1C_2B_2 = \angle B_1CB - \angle CB_1C_2$. But $\angle B_1C_1B = \angle B_1CB$ since BC_1CB_1 is cyclic and $\angle B_2C_1B = \angle OC_1B = 90^\circ - \angle A = 180^\circ - \angle OB_1C = \angle CB_1C_2$. Hence $\angle B_1C_1B_2 = \angle B_1C_2B_2$, so $B_1B_2C_1C_2$ is cyclic, as required. □



Solution 2

The isosceles triangles AB_1B and AC_1C have equal base angles $\angle BAB_1 = \angle C_1AC = \angle A$, so are similar. In particular, $|AB|/|AB_1| = |AC|/|AC_1|$. Since $\angle BAC = \angle B_1AC_1 = \angle A$, it follows that triangles ABC and AB_1C_1 are similar, too. In particular, $\angle CBA = \angle AB_1C_1$.

By construction, lines B_1C_2 and B_2C_1 are the respective perpendicular bisectors of $[AB]$ and $[AC]$, so meet them at their respective midpoints C' and B'. Hence

$$\begin{aligned}\angle B_1C_2B_2 &= \angle C'C_2B = 90^\circ - \angle C_2BC' = 90^\circ - \angle CBA = 90^\circ - \angle AB_1C_1 = 90^\circ - \angle B'B_1C_1 \\ &= \angle B_1C_1B' = \angle B_1C_1B_2.\end{aligned}$$

Hence $B_1C_2C_1B_2$ is cyclic, which completes the proof. □

Solution 3

By construction, lines B_1C_2 and B_2C_1 are the respective perpendicular bisectors of $[AB]$ and $[AC]$, so meet them at their respective midpoints C' and B'. Since $\angle B_1C'C_1 = 90^\circ = \angle C_1B'B_1$, $B_1C_1C'B'$ is cyclic. Together with the fact that $B'C' \parallel BC$ by construction, this implies

$$\angle B_1C_2B_2 = \angle C'C_2B = \angle C_2C'B' = \angle B_1C'B' = \angle B_1C_1B' = \angle B_1C_1B_2,$$

whence $B_1C_2C_1B_2$ is cyclic. This completes the proof. $\square$

Problem 4

A subset A of the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ is called *good* if every integer $n > 0$ has at most one prime divisor p such that $n - p \in A$.

(a) Show that the set $S = \{0, 1, 4, 9, \dots\}$ of perfect squares is good.

(b) Find an infinite good set disjoint from S.

(Two sets are *disjoint* if they have no common elements.)

Solution 1

(a) Suppose to the contrary that S is not good, so there exists $n \in \mathbb{N}$ with two different prime factors $p \neq q$ such that $n - p$, $n - q$ are perfect squares. In particular n is not prime. Write $n - p = m^2$, for some $m \in \mathbb{N}$. As $p \mid n$, it follows that $p \mid m$ and hence $p^2 \mid m^2$ since p is prime. Hence there exists $k \in \mathbb{N}$ such that $n - p = p^2k^2$. Similarly, there exists $\ell \in \mathbb{N}$ such that $n - q = q^2\ell^2$. We observe that $k, \ell \neq 0$ since n is not prime.

Now we have $p - q = (n - q) - (n - p) = (\ell q - kp)(\ell q + kp)$. Since $p - q \neq 0$, $\ell q - kp \neq 0$, and hence $|p - q| = |kp - \ell q||kp + \ell q| \geqslant |kp + \ell q| = kp + \ell q$. This is a contradiction however, because, since $k, \ell \neq 0$, it is clear that $kp + \ell q \geqslant p + q > |p - q|$. Hence S is good. □

(b) Let q be a prime, and let $Q = \{q, q^3, q^5, \dots\}$ be the (infinite) set of odd powers of q, which is disjoint from S. We claim that Q is good. Indeed, let $n \in \mathbb{N}$, and let $p \mid n$ be a prime such that $n - p \in Q$, i.e. $n - p = q^{2k+1}$ for some $k \in \mathbb{N}$. Then $p \mid n - p$, so $p \mid q^{2k+1}$, and hence $p = q$. Thus Q is good. □

Solution 2

(a) Let $p \mid n$ be a prime such that $n - p = p(n/p - 1) = m^2$, for some $m \in \mathbb{N}$. Since $p \mid m^2$ and p is prime, $p^2 \mid m^2$, and hence $p \mid n/p - 1 < n/p$, so $p < \sqrt{n}$.

Now suppose to the contrary that S is not good, so there are primes $p_1 > p_2$ dividing n such that $n - p_1 < n - p_2$ are perfect squares. Then

$$n - p_2 \geqslant \left(\sqrt{n - p_1} + 1\right)^2 > n - p_1 + 2\sqrt{n - p_1} \quad \Longrightarrow \quad p_1 > p_2 + 2\sqrt{n - p_1} \geqslant 2 + 2\sqrt{n - p_1}.$$

The last condition implies that $p_1 > 2\sqrt{n - 1}$. But $p_1 < \sqrt{n}$ by the first part, so $\sqrt{n} > 2\sqrt{n - 1}$, which is a contradiction for $n > 1$; the cases $n = 0$ and $n = 1$ are trivial. Thus S is good. □

(b) We claim that the infinite set $P = \{3, 5, 7, 11, \dots\}$ of odd primes, which is disjoint from S, is good. Indeed, let $n \in \mathbb{N}$ and let $p \mid n$ be a prime such that $n - p = q$, for some odd prime q. Then $p \mid n - p$, so $p \mid q$, i.e. $p = q$, and hence $n = 2q$. Since q is the only odd prime divisor of $n = 2q$, P is good. □

The set $P' = \{2, 3, 5, 7, 11, \dots\}$ of all primes is also good. The proof is similar: let $n \in \mathbb{N}$ and let $p \mid n$ be a prime such that $n - p = q$, for some prime q. Then $p \mid n - p$, so $p \mid q$, i.e. $p = q$, and hence $n = 2q$. If $q = 2$, then 2 is the only prime divisor of n; if $q \neq 2$, then the only prime divisor of n, apart from q, is 2. However, $n - 2 = 2(q - 1) \notin P'$ since $q - 1 > 1$. Hence P' is good. □

Solution 3

(a) Suppose to the contrary that S is not good, so there exists $n \in \mathbb{N}$ with two different prime factors $p \neq q$ such that $n-p$, $n-q$ are perfect squares. Write $n-p=m^2$, for some $m \in \mathbb{N}$. As $p \mid n$, it follows that $p \mid m$ and hence $p^2 \mid m^2$ since p is prime. Hence there exists $k \in \mathbb{N}$ such that $n-p=p^2k^2$. Similarly, there exists $\ell \in \mathbb{N}$ such that $n-q=q^2\ell^2$. By construction, n is not prime, so $n-p, n-q \neq 0$, whence $k, \ell \geqslant 1$.

Hence $p^2k^2+p=q^2\ell^2+q$. Hence $p^2k^2 < p^2k^2+p=q^2\ell^2+q<q^2\ell^2+2q\ell+1=(q\ell+1)^2$. Similarly, $q^2\ell^2<(pk+1)^2$, whence $q\ell-1<pk<q\ell+1$. It follows that $pk=q\ell$, so $p^2k^2+p=q^2\ell^2+q$ yields the contradiction $p=q$. Hence S is good. □

(b) Let A be a finite good set such that $0 \notin A$, and let $m=\max A$. Let $a \geqslant 2m+1$ be an integer. We claim that $A'=A \cup \{a\}$ is good. Indeed, suppose to the contrary that there exist $n \in \mathbb{N}$ and primes $p, q \mid n$ with $p \neq q$ such that $n-p, n-q \in A'$. If $n<a$, then $n-p, n-q \in A$, which is a contradiction because A is good. Hence $n \geqslant a$. Now $p \mid n-p$, so $n-p \geqslant p$ since $0 \notin A'$. Thus $p \leqslant n/2$ and hence $n-p \geqslant n/2 \geqslant a/2 > m$. Similarly, $n-q>m$. It follows that $n-p=n-q=a$, which implies the contradiction $p=q$. Hence A' is good.

Now it is clear that any singleton set is good: indeed, if $A=\{a\}$, and $n \in \mathbb{N}$ has prime divisors p, q such that $n-p, n-q \in A$, then $n-p=a=n-q$, so $p=q$. Starting from the singleton $T_1=\{2\}$, we use the above construction to obtain, iteratively, good sets $T_2, T_3, \ldots$ of $2, 3, \ldots$ elements. It is clearly possibly to ensure that they are each disjoint from S by not adding a perfect square at any stage. Then $T=T_1 \cup T_2 \cup \cdots$ is an infinite good set disjoint from S. □