6th Benelux Mathematical Olympiad

Brugge, May 2–4 2014

1. Find the smallest possible value of the expression

$$\left\lfloor \frac{a+b+c}{d} \right\rfloor + \left\lfloor \frac{b+c+d}{a} \right\rfloor + \left\lfloor \frac{c+d+a}{b} \right\rfloor + \left\lfloor \frac{d+a+b}{c} \right\rfloor,$$

in which a, b, c and d vary over the set of positive integers.

(Here $\lfloor x \rfloor$ denotes the biggest integer which is smaller than or equal to x.)

2. Let $k \geq 1$ be an integer.

We consider $4k$ chips, $2k$ of which are red and $2k$ of which are blue. A sequence of those $4k$ chips can be transformed into another sequence by a so-called *move*, consisting of interchanging a number (possibly one) of consecutive red chips with an equal number of consecutive blue chips. For example, we can move from $r\underline{bb}br\underline{rr}b$ to $r\underline{rr}br\underline{bb}b$ where r denotes a red chip and b denotes a blue chip.

Determine the smallest number n (as a function of k) such that starting from any initial sequence of the $4k$ chips, we need at most n moves to reach the state in which the first $2k$ chips are red.

3. Find all integers $n \geq 2$ with the following property:

> for each pair of positive divisors $k, \ell < n$ of n, at least one of the numbers $2k - \ell$ and $2\ell - k$ is a (not necessarily positive) divisor of n as well.

4. Let $ABCD$ be a square. Consider a variable point P inside the square for which $\angle BAP \geq 60°$. Let Q be the intersection of the line AD and the perpendicular to BP in P. Let R be the intersection of the line BQ and the perpendicular to BP from C.

(a) Prove that $|BP| \geq |BR|$.

(b) For which point(s) P does the inequality in (a) become an equality?

Language: English

Time available: 4.5 hours

Each problem is worth 7 points