5th Benelux Mathematical Olympiad

Dordrecht, 26–28 April 2013

Solutions

Problem 1. Let $n \geqslant 3$ be an integer. A frog is to jump along the real axis, starting at the point 0 and making n jumps: one of length 1, one of length 2, ..., one of length n. It may perform these n jumps in any order. If at some point the frog is sitting on a number $a \leqslant 0$, its next jump must be to the right (towards the positive numbers). If at some point the frog is sitting on a number $a > 0$, its next jump must be to the left (towards the negative numbers). Find the largest positive integer k for which the frog can perform its jumps in such an order that it never lands on any of the numbers 1, 2, ..., k.

Solution. We claim that the largest positive integer k with the given property is $\lfloor \frac{n-1}{2} \rfloor$, where $\lfloor x \rfloor$ is by definition the largest integer not exceeding x.

Consider a sequence of n jumps of length 1,2, ... n such that the frog never lands on any of the numbers 1, 2, ..., k, where $k \geqslant 1$. Note that we must have $k < n$ in order for the frog to be able to make its first jump. As the frog jumps to the right only if it is in a number $a \leqslant 0$, and the largest jump has length n, it is impossible to reach numbers greater than n. On the other hand, suppose the frog is in a number $a > 0$, then it must even be in a number $a \geqslant k+1$, since it is not allowed to hit the numbers 1, 2, ..., k. So the frog jumps to the left only if it is in a number $a \geqslant k+1$, and therefore it is impossible to reach numbers less than $(k+1) - n = k - n + 1$. This means the frog only possibly lands on the numbers i satisfying

$$k - n + 1 \leqslant i \leqslant 0 \quad \text{or} \quad k + 1 \leqslant i \leqslant n. \tag{1}$$

When performing a jump of length k, the frog has to remain at either side of the numbers 1, 2, ..., k. Indeed, jumping over 1, 2, ..., k requires a jump of at least length $k+1$. In case it starts at a number $a > 0$ (in fact $k + 1 \leqslant a \leqslant n$), it lands in $a - k$ and we must also have $a - k \geqslant k + 1$. So $2k + 1 \leqslant a \leqslant n$, therefore $2k + 1 \leqslant n$. In case it starts at a number $a \leqslant 0$ (in fact $k - n + 1 \leqslant a \leqslant 0$), it lands in $a + k$ and we must also have $a + k \leqslant 0$. Adding k to both sides of $k - n + 1 \leqslant a$, we obtain $2k - n + 1 \leqslant a + k \leqslant 0$, so in this case we have $2k + 1 \leqslant n$ as well. We conclude that $k \leqslant \frac{n-1}{2}$. Since k is integer, we even have $k \leqslant \lfloor \frac{n-1}{2} \rfloor$.

Next we prove that this upperbound is sharp: for $k = \lfloor \frac{n-1}{2} \rfloor$ the frog really can perform its jumps in such an order that it never lands on any of the numbers 1, 2, ..., k.

Suppose n is odd, then $\frac{n-1}{2}$ is an integer and we have $k = \frac{n-1}{2}$, so $n = 2k + 1$. We claim that when the frog performs the jumps of length 1, ..., $2k + 1$ in the following order, it does never land on 1, 2, ..., k: it starts with a jump of length $k + 1$, then it performs two jumps, one of length $k + 2$ followed by one of length 1, next two jumps of length $k + 3$ and 2, ..., next two jumps of length $k + (i + 1)$ and i, ..., and finally two jumps of length $k + (k + 1)$ and k. In this order of the jumps every length between 1 and $n = 2k + 1$ does occur: it performs a pair of jumps for $1 \leqslant i \leqslant k$, which are the jumps of length 1, 2, ..., k and the jumps of length $k + 2$, $k + 3$, ..., $2k + 1$, and it starts with the jump of length $k + 1$.

We now prove the correctness of this jumping scheme. After the first jump the frog lands in $k + 1 > k$. Now suppose the frog is in 0 or $k + 1$ and is about to perform the pair of jumps of length $k + (i + 1)$ and i. Starting from 0, it lands in $k + (i + 1) > k$, after which it lands in $(k + i + 1) - i = k + 1 > k$. If on the contrary it starts in $k + 1$, it lands in $(k + 1) - (k + (i + 1)) = -i < 1$, after which it lands in $(-i) + i = 0$. We see that, starting in 0, the frog lands in $k + 1$ after the pair of jumps, while starting in $k + 1$ the frog lands in 0, while in both cases the jumps do not touch 1, 2, ... k. This proves the correctness of its series of jumps. As the frog (after its first jump) alters between $k + 1$ and 0 exactly k times, for odd k it will end up in 0, while for even k it will end up in $k + 1$.

Suppose n is even, then $\frac{n-1}{2}$ is not an integer and we have $k = \frac{n-1}{2} - \frac{1}{2} = \frac{n-2}{2}$, so $n = 2k+2$. Let the frog firstly perform the same series of jumps as in the previous case; they still do not touch 1, 2, ..., k. Now let the frog make a final extra jump of length $2k + 2$. It will land in $0 + (2k + 2) = 2k + 2 > k$ if k is odd, or in $(k + 1) - (2k + 2) = -k - 1 < 1$ if k is even, and its series of jumps is correct again.

We conclude that the largest positive integer k with the given property is $\lfloor \frac{n-1}{2} \rfloor$. $\square$

Problem 2. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + y \leqslant f(f(f(x))) \tag{2}$$

holds for all $x, y \in \mathbb{R}$.

Solution. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying the given inequality (2). Writing z for $x+y$, we find that $f(z) + (z-x) \leqslant f(f(f(x)))$, or equivalently

$$f(z) + z \leqslant f(f(f(x))) + x \tag{3}$$

for all $x, z \in \mathbb{R}$. Substituting $z = f(f(x))$ yields $f(f(f(x))) + f(f(x)) \leqslant f(f(f(x))) + x$, from which we see that

$$f(f(x)) \leqslant x \tag{4}$$

for all $x \in \mathbb{R}$. Substituting $f(x)$ for x we get $f(f(f(x))) \leqslant f(x)$, which combined with (3) gives $f(z) + z \leqslant f(f(f(x))) + x \leqslant f(x) + x$. So

$$f(z) + z \leqslant f(x) + x \tag{5}$$

for all $x, z \in \mathbb{R}$. By symmetry we see that we also have $f(x) + x \leqslant f(z) + z$, from which we conclude that in fact we even have

$$f(z) + z = f(x) + x \tag{6}$$

for all $x, z \in \mathbb{R}$. So $f(z) + z = f(0) + 0$ for all $z \in \mathbb{R}$, and we conclude that $f(z) = c - z$ for some $c \in \mathbb{R}$.

Now we check whether all functions of this form satisfy the given inequality. Let $c \in \mathbb{R}$ be given and consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(z) = c - z$ for all $z \in \mathbb{R}$. Note that $f(f(z)) = c - (c - z) = z$ for all $z \in \mathbb{R}$. For the lefthand side of (2) we find

$$f(x+y) + y = (c - (x+y)) + y = c - x,$$

while the righthand side reads

$$f(f(f(x))) = f(x) = c - x.$$

We see that inequality (2) holds; in fact we even have equality here.

We conclude that the solutions to (2) are given by the functions $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(z) = c - z$ for all $z \in \mathbb{R}$, where c is an arbitrary real constant. $\square$

Problem 3. Let $\triangle ABC$ be a triangle with circumcircle Γ, and let I be the center of the incircle of $\triangle ABC$. The lines AI, BI and CI intersect Γ in $D \neq A$, $E \neq B$ and $F \neq C$. The tangent lines to Γ in F, D and E intersect the lines AI, BI and CI in R, S and T, respectively. Prove that

$$|AR| \cdot |BS| \cdot |CT| = |ID| \cdot |IE| \cdot |IF|. \tag{7}$$

Solution. We first prove that $|DB| = |DI|$. (This may also be claimed by referring to the lemma that D is the centre of the circumcircle of $BICI_a$.) By the constant angle theorem and the fact that AD and BE are angle bisectors of triangle ABC, we see that

$$\angle DBI = \angle DBC + \angle CBI = \angle DAC + \angle CBE = \angle DAB + \angle ABE,$$

while

$$\angle DIB = 180^\circ - \angle AIB = \angle IAB + \angle ABI = \angle DAB + \angle ABE.$$

So $\triangle BDI$ has equal angles $\angle DBI = \angle DIB$, so $|DB| = |DI|$. This proves our claim. We similarly deduce that $|EC| = |EI|$ and $|FA| = |FI|$.

Rewriting (7) into $\frac{|AR|}{|IF|} \cdot \frac{|BS|}{|ID|} \cdot \frac{|CT|}{|IE|} = 1$, we see that it suffices to prove that

$$\frac{|AR|}{|AF|} \cdot \frac{|BS|}{|BD|} \cdot \frac{|CT|}{|CE|} = 1. \tag{8}$$

We now prove by angle chasing that $\triangle RFA \sim \triangle ACI$. As RF is tangent to the circumcircle of $\triangle AFC$, we obtain (using also that CF is angle bisector of $\angle ACB$)

$$\angle RFA = \angle FCA = \angle ICA.$$

Moreover, from $|FA| = |FI|$ we deduce that $\angle FAI = \angle FIA$, so

$$\angle FAR = 180^\circ - \angle FAI = 180^\circ - \angle FIA = \angle CIA.$$

This proves our similarity, which entails that $\frac{|AR|}{|AF|} = \frac{|IA|}{|IC|}$. In the same way we deduce that $\frac{|BS|}{|BD|} = \frac{|IB|}{|IA|}$ and $\frac{|CT|}{|CE|} = \frac{|IC|}{|IB|}$. By these equal ratios we know that

$$\frac{|AR|}{|AF|} \cdot \frac{|BS|}{|BD|} \cdot \frac{|CT|}{|CE|} = \frac{|IA|}{|IC|} \cdot \frac{|IB|}{|IA|} \cdot \frac{|IC|}{|IB|} = 1,$$

which proves (8), as required. $\square$

Problem 4.

a) Find all positive integers g with the following property: for each odd prime number p there exists a positive integer n such that p divides the two integers

$$g^n - n \quad \text{and} \quad g^{n+1} - (n+1).$$

b) Find all positive integers g with the following property: for each odd prime number p there exists a positive integer n such that p divides the two integers

$$g^n - n^2 \quad \text{and} \quad g^{n+1} - (n+1)^2.$$

Solution.

a) Let g be a positive integer with the given property. So for each odd prime number p there exists a positive integer n such that $p \mid g^n - n$ and $p \mid g^{n+1} - (n+1)$.

If g has an odd prime factor p, then from $p \mid g^n - n$ it follows that $p \mid n$, while from $p \mid g^{n+1} - (n+1)$ we deduce that $p \mid n+1$. But p cannot divide both n and $n+1$; contradiction. So g is a power of 2: $g = 2^k$ for some $k \geqslant 0$.

If $g = 2^0 = 1$, then $p \mid 1 - n$ and $p \mid 1 - (n+1)$, which is again a contradiction.

Suppose $k \geqslant 2$. Then $g - 1$ has an odd prime factor p, therefore $g \equiv 1 \pmod p$ so $0 \equiv g^n - n \equiv 1 - n \pmod p$ and $0 \equiv g^{n+1} - (n+1) \equiv 1 - (n+1) \pmod p$, which is again a contradiction.

Now we prove that $g = 2^1 = 2$ does satisfy the condition. Let a prime $p > 2$ be given. Choose $n = (p-1)^2$, then we have $n \equiv (-1)^2 = 1 \pmod p$. By Fermat's little theorem (using $\gcd(2, p) = 1$) we know that $2^{p-1} \equiv 1 \pmod p$, so

$$2^n = 2^{(p-1)^2} = (2^{p-1})^{p-1} \equiv 1 \equiv n \pmod p.$$

Multiplying both sides by 2, we see that also

$$2^{n+1} \equiv 2n = n + n \equiv n + 1 \pmod p.$$

We conclude that only $g = 2$ has the given property.

b) Let g be a positive integer with the given property. So for each odd prime number p there exists a positive integer n such that $p \mid g^n - n^2$ and $p \mid g^{n+1} - (n+1)^2$.

If g has an odd prime factor p, then from $p \mid g^n - n^2$ it follows that $p \mid n^2$, so also $p \mid n$, while from $p \mid g^{n+1} - (n+1)^2$ we deduce that $p \mid (n+1)^2$, so also $p \mid n+1$.

But p cannot divide both n and $n+1$; contradiction. So g is a power of 2: $g = 2^k$ for some $k \geqslant 0$.

If $g = 2^0 = 1$, then for any odd prime p we have $p \mid 1 - n^2 = (1-n)(1+n)$ and $p \mid 1 - (n+1)^2 = (1-(n+1))(1+(n+1))$. Now take $p = 5$. The first statement says that $n \equiv 1$ or $n \equiv -1 \equiv 4 \pmod 5$, and the second that $n \equiv 0$ or $n \equiv -2 \equiv 3 \pmod 5$. But this yields a contradiction.

If $g = 2^1 = 2$, then for any odd prime p we have $p \mid 2^n - n^2$ and $p \mid 2^{n+1} - (n+1)^2$. Now take $p = 3$. As $3 \nmid 2^n$ and $3 \nmid 2^{n+1}$, we know that $3 \nmid n^2$ and $3 \nmid (n+1)^2$. So these two squares must be 1 modulo 3 (as 2 can never be a square modulo 3). Therefore also 2^n and 2^{n+1} must be 1 modulo 3, which gives $2 \cdot 1 \equiv 2 \cdot 2^n = 2^{n+1} \equiv 1 \pmod 3$; contradiction.

Now suppose $k \geqslant 2$. Then $g - 1$ has an odd prime factor p, therefore $g \equiv 1 \pmod p$ so $0 \equiv g^n - n^2 \equiv 1 - n^2 = (1-n)(1+n) \pmod p$ and $0 \equiv g^{n+1} - (n+1)^2 \equiv 1 - (n+1)^2 = (1-(n+1))(1+(n+1)) \pmod p$. Suppose $p \geqslant 5$. The first statement says that $n \equiv 1$ or $n \equiv -1 \pmod p$, and the second that $n \equiv 0$ or $n \equiv -2 \pmod p$. But n can only be congruent to at most one of the numbers -2, -1, 0 and 1, since $p \geqslant 5$; contradiction. We conclude that $p = 3$, so $g - 1$ contains only prime factors 3. Hence $2^k - 1 = 3^\ell$ for some $\ell > 0$. We see that $2^k - 1 \equiv (-1)^k - 1 \pmod 3$, while $3^\ell \equiv 0 \pmod 3$. So k has to be even, say $k = 2m$, and our equation becomes $2^{2m} - 1 = 3^\ell$, or equivalently $(2^m - 1)(2^m + 1) = 3^\ell$. Not both factors on the left-hand side can be divisible by 3, so $2^m - 1 = 1$ and $2^m + 1 = 3^\ell$, so $m = 1$. Hence $g = 2^2 = 4$.

Now we show that $g = 4$ does have the given property. For this we use that $g = 2$ is a solution to part (a): for any odd prime p there exists a positive integer n such that

$$n \equiv 2^n \pmod p \quad \text{and} \quad n + 1 \equiv 2^{n+1} \pmod p.$$

Taking the square of both congruences, we obtain

$$n^2 \equiv (2^n)^2 = (2^2)^n = 4^n \pmod p$$

and

$$(n+1)^2 \equiv (2^{n+1})^2 = (2^2)^{n+1} = 4^{n+1} \pmod p,$$

as desired.

We conclude that only $g = 4$ has the given property.

□